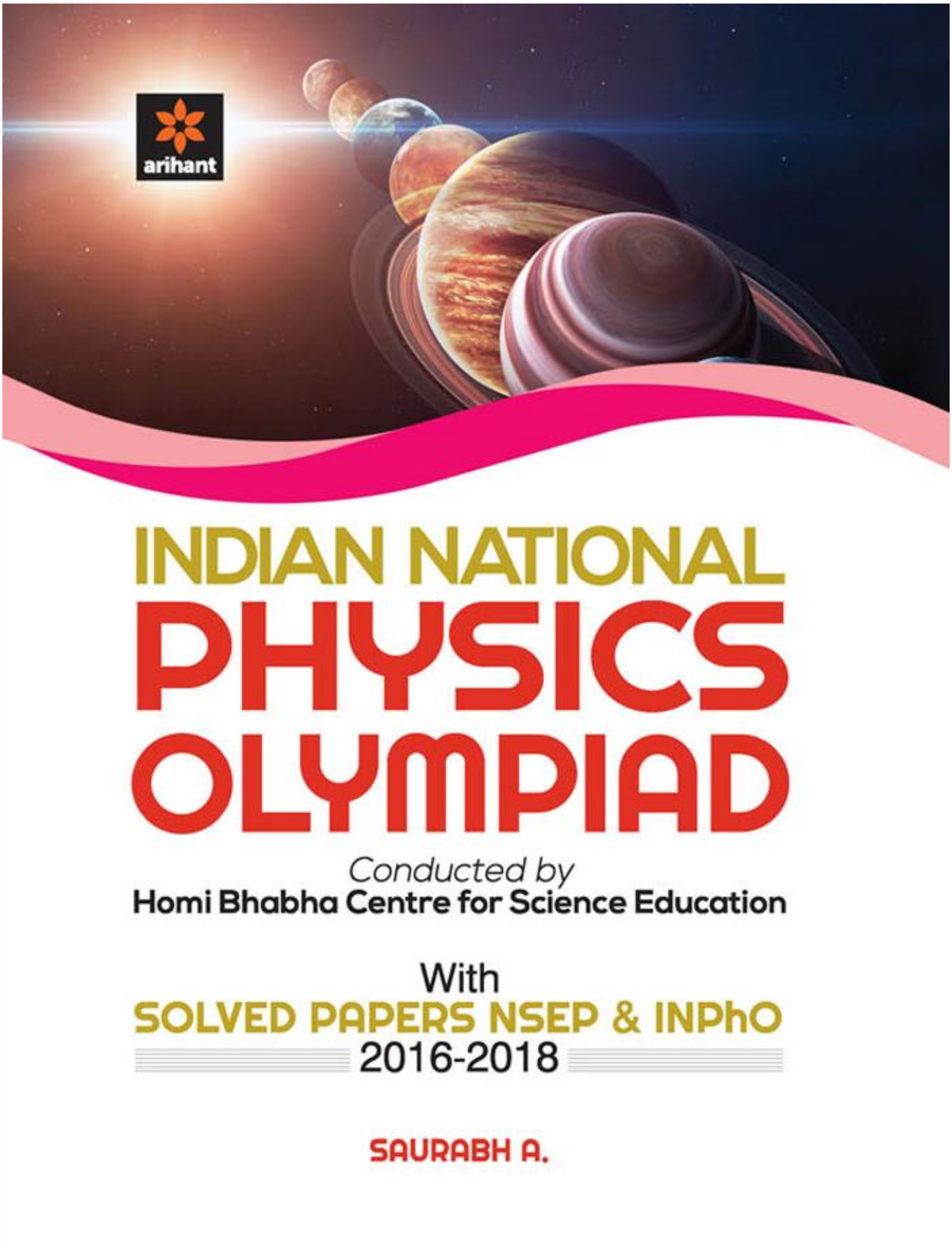

arihant
INDIAN NATIONAL
PHYSICS
OLYMPIAD
Conducted by
Homi Bhabha Centre for Science Education
With
SOLVED PAPERS NSEP & INPhO
2016-2018
SAURABH A.

# INDIAN NATIONAL PHYSICS OLYMPIAD

**Conducted by Homi Bhabha Centre for Science Education**

# First & Foremost

Since the year 1989 when India started participating in the Physics Olympiads, the interest of the school students in the country has tremendously increased in India National Physics Olympiad (INPhO). Now, we find a number of really talented students in almost all the prestigious schools in India who are excited about this mega event and sincerely desire to participate in it, win a handful of medals and make the country proud. We just need to educate them about the competition, and provide them the relevant study material.

The Physics Olympiad tests the participant's level of mastery of the methods of Physics and the strategizing and tactical skills in plenty. The Olympiad is an open challenge to all those who love the problem solving.

This book has been written, keeping in mind the orientation required on the parts of students to face the Olympiads at national or regional level. This book has designed to give the student an insight and proficiency into almost all the areas of Physics. Exhaustive theory has been provided of selected and relevant chapters to clarify the basic concepts. Problems from recently held Olympiads have been given to increase awareness of what to expect in the event.

REVISED EDITION OF THIS BOOK HAS

1. Complete theory with support of good number of solved examples and exactly on the pattern and level of Indian National Physics Olympiads.
2. Each chapter has two level exercises divided according to NSEP and INPhO (Indian National Physics Olympiad)
3. Solutions have been provided for selected questions.

First of all, I would like to thank Mr Deepesh Jain ,Director, Arihant Group the man with a distinct vision, for the idea to write this book, and then bringing it to reality. I am also thankful to my colleagues and students for the moral support they provided. I take this an opportunity to thank Sunil Chugh, Director, HMA, for the inspiration to write the book of this nature, and Sumit Malviya for the assistance he provided in the preparation of the manuscript.

It is hoped, this book will charge you up for the Olympiad juggernaut. I have tried my best to keep this book error-free. However, if there any error is left I request the readers to bring forward to my notice. Suggestions for the further improvement of the book are welcome.

*With best wishes*

Saurabh A.

# CONTENTS

# Introduction
# PHYSICS OLYMPIADS

## ABOUT THE EXAM & HOW TO SUCCEED IN IT ?

### National Physics Olympiad

The National Physics Olympiad (INPhO) is an annual physics competition for high school students. It is one of the International Science Olympiads. The first IPhO was held in Warsaw, Poland in 1967. India started it participating in International Physics Olympiad from 1998. In 2008, there were 80 countries which sent their delegates to participate in the Olympiad.

Each national delegation is made up of five student competitors plus two leaders, selected on a national level. Observers may also accompany a national team. The students compete as individuals, and must sit for intensive theoretical and laboratory examinations. For their efforts the students can be awarded gold, silver, or bronze medals or an honourable mention.

The theory test consists of three questions and that have to be done in 5 hours. Usually these questions involve more than one part. The practical examination may consist of one laboratory examination of five hours or two which together take up the full 5 hours.

The competition is held for two days. The first day is given three theoretical problems (three problems involving at least four areas of physics taught in secondary schools, total number of marks is 30). Another day is given some experimental problems (one or two problems, total number of marks 20). These two days are separated by at least one day of rest. On both occasions the time alloted for solving the problems is 5 hours. Each team consists of students from general or technical secondary schools (not colleges or universities). Typically each team consists of five students (pupils) and two supervisors.

The minimal scores required for Olympic Medals and Honourable Mentions are chosen by the organisers according to the following rules:

- A Gold Medal should be awarded to the top 6% of the participants.
- A Silver Medal should be awarded to the top 18%.
- A Bronze Medal should be awarded to the top 36%.
- An Honourable Mention should be awarded to the top 60%.
- All other participants receive certificates of participation.
- The participant with the highest score (Absolute Winner) receives a special prize, in addition to a Gold Medal.

As in every International Physics Olympiad, the IPhO provides a unique opportunity for high school students to demonstrate their abilities in physics, exchange experiences and build cross-cultural contacts. Participants will experience the opportunity to make new and lasting friendships with peers from all over the world; visit archeological sites full of ancient history; admire the beautiful landscape, and to attend talks by world renown scientists regarding interesting research in physics.

## Indian National Physics Olympiad

Indian National Physics Olympiad is held as a part of the selection process of International Physics Olympiad. The first part of the selection procedure which involves written examination, is held at the end of January or beginning of February. Indian Association of Physics Teachers (IAPT) is responsible for conducting the examination for selection.

## Stages

**Stage I** National Standard Examination in Physics (NSEP)

Before they become a part of Indian National Physics Olympiad, the students of Classes XI and XII need to qualify for in the National Standard Examination in Physics which is held in the month of November prior to Physics Olympiads. Among the 30,000 students who appear for this examination only 1% qualify for the Indian National Physics Olympiads.

**Stage II** Indian National Physics Olympiad (INPhO)

Based on performance in NSEP, the top 300 students in order of merit qualify to appear for the second stage of the Olympiad programme (INPhO). In case there is a tie at the last position, all students with the same marks qualify for the INPhO.

All students who qualify to appear for the INPhO get a certificate of merit from IAPT. INPhO is organised by HBCSE at about 15 centres in the country.

**Stage III** Orientation Cum Selection Camp (OCSC) in Physics

On the basis of performance in INPhO, the top 35 students in the merit list are selected for Stage III, Orientation Cumtional Physics Olympiad (IPhO), provided they satisfy required criteria such as age limit, holding valid Indian passport, medical fitness, parental consent, etc.

**Stage IV** Pre-departure Training (PDT) Camp for IPhO

The selected 5 member Indian team undergoes a rigorous training programme at HBCSE in theory and experiments. Special laboratories have been developed at HBCSE for the purpose of experimental training. Resource persons from HBCSE and different institutions across the country train the students.

**Stage V** Participation in International Physics Olympiad (IPhO)

The 5 member student team, 2 teacher leaders and 1 scientific observer constitute the delegation to represent India at the International Physics Olympiad (IPhO).

## Career Prospects

The Olympiad gives the students a good exposure to Physics and makes them aware of the latest development in the area of science. The students gain immense confidence after participating in this contest, and their horizon of knowledge broadens considerably after that. Moreover, it adds to their curriculum vitae and is viewed as an achievement. Those students who qualify for the training camp at Mumbai are assured of direct admission to BSc Physics programme at Chennai Physics Institute.

## Craze of The Examination

The students of Class XI and XII form a part of Indian and International Olympiad. These contests are responsible for generating interest in Physics among the students. The complex nature of problems presented in the examination and difficult procedure of selection makes participating in this examination an enviable position. Since the questions are based on the Class XII level, most of the candidates are familiar with the principles and they try to solve the questions. They consult books which have complex questions based on Physics and its properties and solve the questions. Prospect of better future, once their talent is recognised, keeps them interested in the Physics Olympiad.

## General Eligibility

For International Physics Olympiad, the candidates must be pre-university students studying in Class XI or XII of a recognised school in their country. The candidates must be 16-18 years of age.

For the Indian National Physics Olympiad, the candidates must be in the age group of 16-18 years and must have passed through the selection procedure.

## Skill Set Required

The competition in Olympiads is so stiff that only a near perfect candidate with exceptional ability to calculate and decipher the principles of Physics can even participate in it. The level is so high and the problems are so sophisticated that sometimes even the teachers of Physics take time in solving those problems.

The candidates must develop ability to think quickly and substantiate their answers with reasons and explanations. Speed with accuracy while computing the mathematical expressions is essential criterion. The candidates must be well versed in all the fundamentals of Physics and should be able to employ the principles simultaneously. The candidates must possess analytical ability and must be able to comprehend the crux of the problems and give solutions accordingly.

## Difficulty Level

These examinations are highly sophisticated methods of testing the students' ability in Physics. The problems range from course of Class XI to XII and all the basic concepts are also tested thoroughly. The candidates must have a sound knowledge of the subject and must be able to employ at will the various formulae, mix the knowledge of the fundamentals with the advanced level and should be able to express in both written and practical examinations.

Every topic is complex and requires thorough study. Those who are able to crack the Olympiads are easily able to pass the IIT entrance examination.

## Previous Years' Trend Analysis

The questions cover the entire course, there are questions from Heat and Thermodynamics, Relativistic Velocity Transformation and Collision Mechanics.
There are subdivisions and questions are based on various principles.

## How To Prepare for Different Topics In Stipulated Time

The subject Physics is regarded as possessing a highly complex domain. At more advance level, the subject turns inter-disciplinary. Preparation for pure Physics is a challenge. Physics involves computation of problems along the line of Mathematics. The best way to prepare for the Olympiads is go back to Class IX and X and start from there. For the complex problems there are books by Arihant which help in preparing for the examination.

## General Mental Set Up Required For The Examination

The candidates, to participate in this examination, must have the power of concentration and have the ability to sustain stamina for working over a longer duration of time. Since the work involves exercise of mental faculty, it is essential that the candidates remain free of stress and inclined only to think in the direction of doing well in the examination. The candidates must possess self-confidence and be totally involved in the task of preparing for the examination. They must have ability to think creatively and must be familiar with all the basic concepts of Physics.

## Do's and Don't on The Day of Examination

On the day of the examination, the candidates must take care about a few things which are listed below:

- The candidates must reach the venue of the examination at least half an hour before time.
- The candidates must carry their admit cards to the examination hall.
- The candidates must carry their own pens, pencils, erasers, sharpeners and must refrain from borrowing these articles from the other candidates.
- The candidates must abstain from talking to other candidates in the examination hall while the examination is being conducted.
- The candidates must hand over their answer sheets to the invigilator as soon as the stipulated time is over.

## How This Book Is Useful For You

This book is a collection of lessons which contain challenging problems from various National and International Physics Olympiads. The problems are provided with complete solution and sometimes alternate method of solving a problem is also given which is very useful for the candidates. Effort has been put in this book to make it a wholesome and utilitarian self-study guide for preparation for National and International Olympiads. The pattern of the examination has been faithfully maintained to give the students real feel of the examination. Most of the questions figuring in this book are from National and International Olympiads.

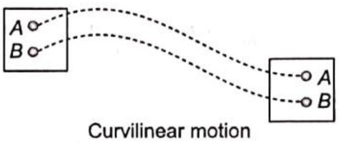
# Unit 1

# Particle Mechanics

## Types of Motion

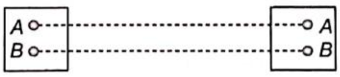
1. **Translatory Motion** When a body moves in such a way that the linear distance covered by each constituent particle of body remains same.

   (a) **Curvilinear Motion** Path is curved.



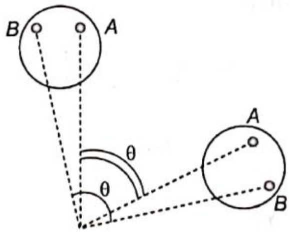

Curvilinear motion

   (b) **Rectilinear Motion** Path is straight.




Rectilinear motion

2. **Rotatory Motion or Rotation** When a body moves in such a way that every constituent particle of body covers same angular displacement.




## Components of Translatory Motion

(a) Displacement ($s$) = Minimum distance between two points.

(b) Velocity $(v) = \dfrac{\text{Displacement}}{\text{Time}}$

(c) Average velocity $\langle v \rangle = \dfrac{\text{Total displacement}}{\text{Total time}}$

(d) Instantaneous velocity ($v_t$)

$$\lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$

$$v_t = \tan\theta$$

(e) Acceleration $$(a) = \frac{v_{\text{final}} - v_{\text{initial}}}{\text{Time}}$$

$$<a> = \frac{\Delta v}{\Delta t}$$

(f) Instantaneous acceleration ($a_t$)

$$\lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$$

$$a_t = \tan\theta$$

## Components of Rotatory Motion

(a) Angular displacement = ($\theta$)

(b) Angular velocity $(\omega) = \frac{\Delta\theta}{\Delta t}$

$$\omega = \frac{\text{Total angular displacement}}{\text{Total time}}$$

(c) Instantaneous angular velocity ($\omega_t$)

$$\omega_t = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

$$\omega_t = \tan\theta$$

(d) Angular acceleration $(\alpha) = \frac{\Delta\omega}{\Delta t}$

(e) Instantaneous angular acceleration ($\alpha_t$)

$$\alpha_t = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$$

$$\alpha_t = \tan\theta$$

## Equations of Motion (Translatory)

1. $v = u \pm at$
2. $v^2 = u^2 \pm 2as$
3. $s = <v> t = \left(\frac{v+u}{2}\right)t$

or $s = ut \pm \frac{1}{2}at^2$

**Note** Equations of motion are valid only for uniformly accelerated motion.

### Special Cases

1. For a particle starting from rest, $u = 0$
2. For a particle stops after sometime, $v = 0$
3. For a particle, starting from a point and returning to the same point, displacement $s = 0$.

## Equations of Rotatory Motion

1. $\omega = \omega_0 + \alpha t$ 2. $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$ 3. $\omega^2 - \omega_0^2 = 2\alpha\theta$

These equations are valid only if angular acceleration of body is uniform.

**Example 1.** *The position of a particle moving along the x-axis depends on time as follows*

$$x = \frac{u}{k}(1 - e^{-kt})$$

*where u and k are constants.*

(*a*) *What is the total displacement of the particle and the distance covered by it?*

(*b*) *How are the velocity and acceleration related to each other?*

**Solution** (a) Total displacement $= x(\infty) - x(0)$

$$\Rightarrow \quad = \frac{u}{k}\left(1 - \frac{1}{e^{\infty}}\right) - \frac{\mu}{k}\left(1 - \frac{1}{e^0}\right) = \frac{u}{k}(1 - 0) - \frac{\mu}{k}(1 - 1) = \frac{u}{k} \text{ unit}$$

Instantaneous velocity $v = \frac{dx}{dt} = \frac{u}{k}[0 - (-k)e^{-kt})] = \frac{u}{k}(ke^{-kt}) = ue^{-kt}$

Putting $v = 0$

$$0 = ue^{-kt} \Rightarrow \frac{1}{e^{kt}} = 0 \Rightarrow e^{kt} = \infty \text{ or } t = \infty$$

This indicate that the velocity will decrease from $u$ to 0 in infinite time, *i.e.*,velocity will not change its direction.

$\therefore$ Distance covered = Displacement covered = $\frac{u}{k}$ unit

(b) Instantaneous velocity, $v = ue^{-kt}$

Instantaneous acceleration, $a = \frac{dv}{dt} = u(-ke^{-kt}) = -k(ue^{-kt})$

$$a = -kv$$

Thus the acceleration at any instant is proportional to the magnitude of velocity and is directed opposite to the velocity.

## Motion Under Gravity

1. Maximum height attained by a particle projected vertically upward with an initial velocity $u$.

$$H = \frac{u^2}{2g}$$

2. Time taken to attain $H_{max}$

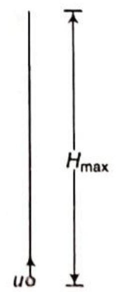
$$t = \frac{u}{g}$$

3. Total time of flight

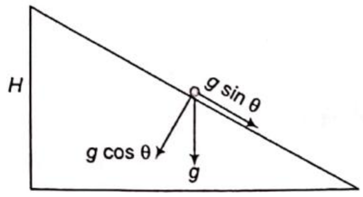
$$T = \frac{2u}{g}$$

4. Velocity at a given height

$$v = \pm\sqrt{u^2 - 2gH}$$



## Motion Along an Inclined Plane




Effective acceleration on the body along the plane $= g\sin\theta$

Time taken in sliding a particle down the whole length of the incline $T = \sqrt{\dfrac{2H}{g}}\ \text{cosec}\theta$

⇒ Velocity acquired by a particle in sliding down an inclined plane is the same as that acquired by a particle falling freely from rest through a distance equal to the height of the inclined plane.

$$v = \sqrt{2gH}$$

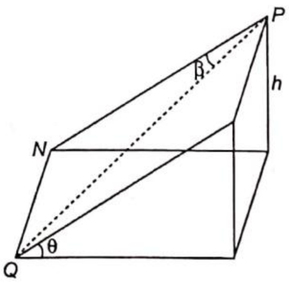
# Illustrative Solved Examples

**Example 2.** *A particle is moving along an expanding spiral in such a manner that the particle's normal acceleration remains constant. How will the linear angular velocity change in the process?*

**Solution** The normal acceleration is $\omega_n = \dfrac{v^2}{R} = \omega^2 R$, where $R$ is the radius of spiral.

Whence the linear velocity grows in proportion to the square root of the curvature radius of spiral, while the angular velocity decreases by same law.

**Example 3.** *Find the time taken by the particle to slide down a height h along the line PQ.*




**Solution** Here $PN$ is the line of greatest slop.

So, component of acceleration acting along $PN = g\sin\theta$

So, component of acceleration acting along $PQ = a_{PQ} = g\sin\theta\sec\beta$

From right angled triangle $PNQ$,

$$\cos\beta = \frac{PN}{PQ}$$

$$PQ = PN\sec\beta$$

Now $u = 0$, $$a = g\sin\theta\cos\beta$$

Using $$s = ut + \frac{1}{2}at^2$$

$$h \operatorname{cosec}\theta \sec\beta = \frac{1}{2}(g\sin\theta\cos\beta)t^2$$

$$t^2 = \frac{2h}{g}(\operatorname{cosec}^2\theta \sec^2\beta)$$

$$t = \sqrt{\frac{2h}{g}}\operatorname{cosec}\theta\sec\beta$$

Since $\sec\beta > 1$ for $0<\beta<90°$

$$t > \sqrt{\frac{2h}{g}}\operatorname{cosec}\theta$$

$t >$ time of sliding along $PN$.

⇒ Time taken to slide down an inclined plane is least along the line of greatest slope.

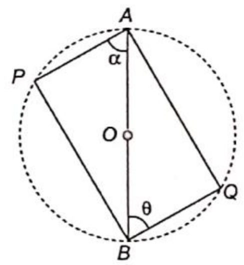
⇒ Line of greatest slope is also reffered as line of quickest descent.

**Example 4.** (*a*) *Show that time taken by a particle to slide from any point on a vertical circle along a smooth chord terminating at the lowest point of the circle is constant.*

(*b*) *Time taken by particle to slide down any chord of smooth vertical circle, starting from rest at highest point is constant.*

**Solution** (a) Consider any chord $QB$ making an angle $\theta$ with the vertical diameter $AB$

For motion of particle along $QB$




$u = 0$, $s = QB$

$a = g\cos\theta$, $t = ?$

Using $s = ut + \frac{1}{2}at^2$

$$QB = \frac{1}{2}g\cos\theta\, t^2$$

$$t^2 = \frac{2\,QB}{g\cos\theta}$$

From right angled triangle $AQB$

$$\cos\theta = \frac{QB}{AB}$$

$$t^2 = \frac{2AB}{g} = \frac{2}{g}(d)$$

$$t = \sqrt{\frac{2d}{g}} = \text{constant.}$$

(b) Let $AP$ be any chord drawn from $A$ on the circle. Making angle $\alpha$ with the vertical.

Angle subtended by chord $AP$ with horizontal $= 90° - \alpha$

For motion of particle from $A$ to $P$

$u = 0$, $a = g\sin(90° - \alpha)$, $s = AP$

Using $s = ut + \frac{1}{2}at^2$

$$AP = \frac{1}{2}g\cos\alpha\, t^2$$

$$t^2 = \frac{2\,AP}{g\cos\alpha}$$

From right angled triangle $APB$

$$\cos\alpha = \frac{AP}{AB}$$

$$t^2 = \frac{2\,AB}{g} \quad \Rightarrow \quad t = \sqrt{\frac{2d}{g}} = \text{constant.}$$

**Example 5.** *If a point moves with constant acceleration, then show that space average of velocity over any distance is* $\frac{2}{3}\left(\frac{u_1^2 + u_1u_2 + u_2^2}{u_1 + u_2}\right)$ *and the time average of velocity is* $\frac{1}{2}(u_1 + u_2)$, *where* $u_1$ *and* $u_2$ *are the initial and final velocities.*

**Solution** Let $v$ be the velocity and $s$ the distance travelled from the starting point at time $t$.

Let $a$ be the whole distance moved and $T$ be the whole time taken, then we have

$$\text{Space average of velocity} = \frac{1}{a}\int_{s=0}^{a} v\,ds$$

and $$\text{Time average of velocity} = \frac{1}{T}\int_{t=0}^{T} v\,dt$$

Let $A$ be the acceleration of the point.

Then from $v^2 = u^2 + 2As$, we get

$$v^2 = u_1^2 + 2\,As \qquad \text{...(i)}$$

and $$u_2^2 = u_1^2 + 2\,Aa \qquad \text{...(ii)}$$

Also $$v = u_1 + At \qquad \text{...(iii)}$$

and $$u_2 = u_1 + AT \qquad \text{...(iv)}$$

$\therefore$ Space average of velocity $= \frac{1}{a}\int_{s=0}^{a} v\,ds$

$$= \frac{1}{a}\int_{s=0}^{a} (u_1^2 + 2\,As)^{1/2}\,ds \qquad \text{[From Eq. (i)]}$$

$$= \frac{1}{a}\left[\frac{2}{3}\frac{(u_1^2 + 2\,As)^{3/2}}{2A}\right]_0^a = \frac{(u_1^2 + 2\,Aa)^{3/2} - (u_1^2)^{3/2}}{3\,Aa}$$

$$= \frac{(u_2^2)^{3/2} - (u_1^2)^{3/2}}{\frac{3}{2}(u_2^2 - u_1^2)} \qquad \left[\because Aa = \frac{u_2^2 - u_1^2}{2}, \text{ From Eq. (ii)}\right]$$

$$= \frac{2}{3}\frac{(u_2^3 - u_1^3)}{(u_2^2 - u_1^2)} = \frac{2\,(u_2^2 + u_1u_2 + u_1^2)}{3\,(u_2 + u_1)}$$

and time average of velocity

$$= \frac{1}{T}\int_{t=0}^{T} v\,dt = \frac{1}{T}\int_0^T (u_1 + At)\,dt \qquad \text{[From Eq. (iii)]}$$

$$= \frac{1}{T}\left[\frac{(u_1 + At)^2}{2\,A}\right]_0^T = \frac{(u_1 + AT)^2 - u_1^2}{2\,AT} = \frac{u_2^2 - u_1^2}{2\,(u_2 - u_1)}$$

$$AT = u_2 - u_1 \qquad \text{[From Eq. (iv)]}$$

$$= \frac{1}{2}(u_1 + u_2)$$

**Example 6.** *A lift ascends with constant acceleration a, then with constant velocity and finally stops under constant retardation a. If the total distance ascended is s and total time occupied is t show that the time during which the lift is ascended with constant velocity is* $\sqrt{(t^2 - 4s/a)}$.

**Solution** The whole journey consists of three parts. Let $v$ be the maximum velocity acquired in 1st part, which will remain uniform in 2nd part and will gradually reduce to zero in 3rd part due to retardation. Since acceleration in the first part and retardation in the 3rd part are equal, therefore the time taken and distance covered in acquiring velocity $v$ in first part from start will be equal to the corresponding time and distance in 3rd part in destroying the velocity $v$. Let distance travelled be $x_1$ and time taken be $t_1$ in each of the parts.

∴ In the first and last parts of motion,

we have $$v = at_1 \quad \text{... (i)}$$

and $$v^2 = 2a x_1 \quad \text{...(ii)}$$

In the second part of motion, distance moved $= s - 2x_1$ and time taken $= t - 2t_1$

∴ $$(s - 2x_1) = v(t - 2t_1) \quad \text{...(iii)}$$

From Eqs. (i), (ii) and (iii), we get

$$s - \frac{v^2}{a} = at_1(t - 2t_1)$$

or $$s - at_1^2 = at_1(t - 2t_1) \quad \text{[From Eq. (i)]}$$

or $$at_1^2 - att_1 + s = 0$$

or $$t_1 = \frac{at \pm \sqrt{a^2t^2 - 4as}}{2a}$$

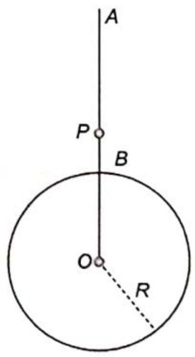
or $$2t_1 = t \pm \sqrt{t^2 - \frac{4s}{a}}$$

or $$t - 2t_1 = \sqrt{t^2 - \frac{4s}{a}}$$

**Example 7.** *If h be the height due to velocity v at the earth's surface, supposing the acceleration due to gravity to be constant, and H the corresponding height when the variation of gravity is taken into account, prove*

$$\frac{1}{h} - \frac{1}{H} = \frac{1}{R}$$

*where R is the radius of the earth.*




**Solution** Supposing the acceleration due to gravity to be constant and equal to $g$, according to the problem we have

$$v^2 = 0 + 2gh \quad \text{or} \quad v^2 = 2gh \quad \text{...(i)}$$

when variation of gravity is taken into account, let $P$ be the position of particle at time $t$, such that $OP = x$

∴ The equation of motion is

$$m\frac{d^2x}{dt^2} = -\frac{m\mu}{x^2}$$

or $$\frac{d^2x}{dt^2} = -\frac{\mu}{x^2} \quad \text{...(ii)}$$

On the surface of the earth $x = R$

∴ $$g = \mu / R^2 \text{ or } \mu = gR^2 \quad \text{...(iii)}$$

From Eqs. (iii) and (ii), we have

$$v\frac{dv}{dx} = -\frac{gR^2}{x^2}$$

$$\frac{d^2x}{dt^2} = v\frac{dv}{dx}$$

Integrating with respect to $x$, we get

$$\frac{1}{2}v^2 = -\frac{gR^2}{x} + C_1 \qquad \text{...(iv)}$$

At the highest point $A$, $v = 0$ and $x = OB + BA = R + H$

$\therefore$ From Eq.(iv), we get $0 = \frac{gR^2}{H+R} + C_1$ or $C_1 = \frac{-gR^2}{H+R}$

$$v^2 = 2gR^2\left[\frac{1}{x} - \frac{1}{H+R}\right]$$

On the surface of the earth, *i.e.*, at $B$, $v = v$ and $x = R$

$\therefore$ $$v^2 = 2gR^2\left[\frac{1}{R} - \frac{1}{H+R}\right] \quad \text{or} \quad 2gh = 2gR^2\left[\frac{H+R-R}{R(H+R)}\right]$$

From Eq.(i) $$v^2 = 2gh$$

or $$h = \frac{RH}{H+R} \quad \text{or} \quad \frac{1}{h} = \frac{H+R}{RH} \quad \text{or} \quad \frac{1}{h} - \frac{1}{H} = \frac{1}{R}.$$

**Example 8.** *A particle starts from rest at a distance a from the centre of force which attracts inversely as the distance. Prove that the time of arriving at the centre is* $a\sqrt{\pi/2\mu}$, *where* $\mu$ *is a constant.*

**Solution** The equation of motion is $$\frac{d^2x}{dt^2} = -\frac{\mu}{x} \qquad \text{...(i)}$$

Integrating Eq. (i) $$\left(\frac{dx}{dt}\right)^2 = -2\mu\log x + C$$

Initially at $x = a$, $$\frac{dx}{dt} = 0 \qquad \text{(Given)}$$

$\therefore$ $$0 = -2\mu\log a + C \quad \text{or} \quad C = 2\mu\log a$$

$\therefore$ $$\left(\frac{dx}{dt}\right)^2 = 2\mu(\log a - \log x)$$

or $$\frac{dx}{dt} = -\sqrt{2\mu\log(a/x)} \qquad \text{...(ii)}$$

The negative sign is due to the fact that the particle is moving towards the centre.

$\therefore$ From Eq. (ii), we get $$dt = \frac{1}{\sqrt{2\mu}}\cdot\frac{dx}{\sqrt{(\log a/x)}}$$

$\therefore$ Required time from $x = a$ to $x = 0$

$$= -\frac{1}{\sqrt{2\mu}}\int_{x=a}^{0}\frac{dx}{\sqrt{(\log a/x)}}$$

$$= \frac{1}{\sqrt{2\mu}}\int_{x=0}^{\infty}\frac{-2aze^{-z^2}dz}{z} \qquad \left[\begin{array}{l}\text{Putting } \sqrt{\log(a/x)} = z \\ \text{or } x = e^{-z^2} \text{ or } dx = 2aze^{-z^2}dz\end{array}\right]$$

$$= \frac{2a}{\sqrt{2\mu}}\int_0^{\infty}e^{-z^2}dz$$

$$= \frac{2a}{\sqrt{2\mu}} \cdot \frac{\sqrt{\pi}}{2} \qquad \left(\because \int_0^{\pi} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}\right)$$

$$= a\sqrt{\frac{\pi}{2\mu}}$$

**Example 9.** *A particle is attracted by a force to a fixed point varying inversely as (distance)$^n$. If the velocity acquired in falling from an infinite distance to a distance a from the centre be equal to the velocity acquired in falling from rest from distance a to distance $\frac{1}{4}a$, prove that $n = 3/2$.*

**Solution** Given that $\frac{d^2x}{dt^2} = -\frac{\mu}{x^n} = -\mu\, x^{-n}$ ...(i)

where $x$ is the distance of the particle from the fixed point at time $t$.

Multiplying both sides by $2(dx/dt)$ and integrating, we get

$$\left(\frac{dx}{dt}\right)^2 = -2\mu\, [x^{-n+1} / (-n+1)] + C$$

or

$$\left(\frac{dx}{dt}\right)^2 = \frac{2\mu}{(n-1)x^{n-1}} + C \qquad \text{...(ii)}$$

If the particle falls from rest at infinity, then at $x = \infty$, $dx/dt = 0$

$\therefore$ $C = 0$

$\therefore$ From Eq. (ii), we get

$$\left(\frac{dx}{dt}\right)^2 = 2\mu / [(n-1)x^{n-1}]$$

$\therefore$ If $v$ be the velocity of the particle at $x = a$, then

$$v^2 = 2\mu / [(n-1)a^{n-1}] \qquad \text{...(iii)}$$

Again, if the particle start from rest from $x = a$, then at $x = a$, $dx/dt = 0$ and from Eq. (ii), we have

$$0 = \frac{2\mu}{(n-1)a^{n-1}} + C \text{ or } C = -\frac{2\mu}{(n-1)a^{n-1}}$$

$\therefore$ From Eq. (ii), we get

$$\left(\frac{dx}{dt}\right)^2 = \frac{2\mu}{n-1}\left[\frac{1}{x^{n-1}} - \frac{1}{a^{n-1}}\right]$$

$\therefore$ If $v_1$ be the velocity of the particle at $x = \frac{1}{4}a$, we have

$$v_1^2 = \frac{2\mu}{(n-1)}\left[\frac{1}{\left(\frac{1}{4}a\right)^{n-1}} - \frac{1}{a^{n-1}}\right] \qquad \text{...(iv)}$$

Now, if $v = v_1$, from Eqs. (iii) and (iv), we get

$$\frac{2\mu}{(n-1)a^{n-1}} = \frac{2\mu}{(n-1)}\left[\frac{4^{n-1}}{a^{n-1}} - \frac{1}{a^{n-1}}\right]$$

$$1 = 4^{n-1} - 1 \quad \text{or} \quad 2 = 4^{n-1}$$

or

$$2 = (2^2)^{n-1} = 2^{2n-2}$$

or

$$2^3 = 2^{2n} \quad \text{or} \quad 2n = 3 \quad \text{or} \quad n = \frac{3}{2}$$

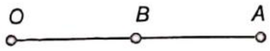
**Example 10.** *If a particle is projected towards the centre of repulsion, the repulsion varying as the distance from the centre which in this case is s with a velocity $s\sqrt{\mu}$, prove that the particle will approach the centre but never reach it.*

**Solution** Let the particle be at $B$ at a distance $x$ from the centre $O$ after time $t$.

Also $$OA = s \text{ (Given)}$$

The force of repulsion means that the acceleration is away from $O$ in the direction $OB$. Also the particle is projected from $A$ towards $O$.

$\therefore$ The equation of motion is $\dfrac{d^2x}{dt^2} = \mu x$ ...(i)




(The acceleration being away from $O$, we have + ve sign)

Integrating Eq. (i), we have $(dx/dt)^2 = \mu x^2 + C$

Given that at $x = s, dx/dt = s\sqrt{\mu}$ $\therefore$ $s^2\mu = s^2\mu + C$ or $C = 0$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2 = \mu x^2 \quad \text{or} \quad \frac{dx}{dt} = -\sqrt{\mu}\, x \qquad \text{...(ii)}$$

(Negative sign has been taken as the particle moves towards $O$)

$$\text{or} \quad dt = -\left(\frac{1}{\sqrt{\mu}}\right)\left(\frac{1}{x}\right)dx$$

$\therefore$ Time to reach the centre $O$ from $A$ (*i.e.*, from $x = s$ to $x = 0$)

$$= -\frac{1}{\sqrt{\mu}}\int_{x=s}^{0}\frac{dx}{x} = \frac{1}{\sqrt{\mu}}[\log x]_0^s = \frac{1}{\sqrt{\mu}}[\log s - \log 0]$$

$$= \frac{1}{\sqrt{\mu}}[\log s - (-\infty)] \qquad (\because \log 0 = -\infty)$$

$$= \infty$$

*i.e.*, the particle will reach the centre after infinite time *i.e.*, the particle will never reach the centre $O$.

**Example 11.** *A particle moves in the curve $y = a \log \sec (x/a)$ in such a way that the tangent to the curve rotates uniformly; prove that the resultant acceleration of the particle varies as the square of the radius of curvature.*

**Solution** The equation of the path is

$$y = a \log \sec \frac{x}{a} \qquad \text{...(i)}$$

As the tangent to the curve rotates uniformly,

$$\text{So,} \quad \frac{d\psi}{dt} = \text{constant} = \omega \text{ (say)} \qquad \text{...(ii)}$$

Differentiating Eq. (i), we get

$$\frac{dy}{dx} = a\,\frac{1}{\sec (x/a)}.\sec\frac{x}{a}\tan\frac{x}{a}.\frac{1}{a}$$

$$\text{or} \quad \frac{dy}{dx} = \tan\left(\frac{x}{a}\right) \qquad \text{...(iii)}$$

$$\text{Differentiating again} \quad \frac{d^2y}{dx^2} = \frac{1}{a}\sec^2\left(\frac{x}{a}\right)$$

$$\text{The radius of curvature} = \frac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2} = \frac{[1 + \tan^2 (x/a)]^{3/2}}{\frac{1}{a}\sec^2 (x/a)} = a \sec\frac{x}{a} = \rho \qquad \text{...(iv)}$$

Also, from Eq. (iii), we get

$$\tan\psi = \frac{dy}{dx} = \tan(x/a) \quad \text{or} \quad \psi = x/a \quad \text{or} \quad x = a\psi$$

Differentiating with respect to $t$, we get

$$\frac{dx}{dt} = a\frac{d\psi}{dt} = a\omega \qquad \text{[From Eq. (ii)]}$$

$$\therefore \quad \frac{d^2x}{dt^2} = 0 \qquad (\because a\omega = \text{constant})$$

Also $$\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt} = \left[\tan\left(\frac{x}{a}\right)\right]a\omega$$

$$\therefore \quad \frac{d^2y}{dt^2} = a\omega\,.\sec^2\left(\frac{x}{a}\right).\frac{1}{a}.\frac{dx}{dt} = a\omega^2\sec^2\left(\frac{x}{a}\right)$$

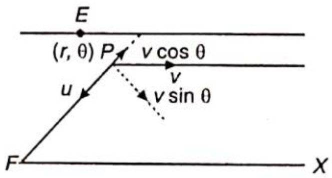
Now resultant acceleration $= \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2} = \sqrt{0 + a^2\omega^4\sec^4(x/a)}$

$$= a\omega^2\sec^2(x/a) = (\omega^2/a)\rho^2 \qquad \text{[From Eq. (iv)]}$$

Hence, resultant acceleration varies as square of radius of curvature.

**Example 12.** *A boat which is rowed with constant velocity u starts from a point E on the bank of a river which flows with a constant velocity v, and it points always towards a point F on the other bank exactly opposite to E. Find the equation of the path of the boat. If $v = u$, show that the path is a parabola whose focus is F.*

**Solution** Let $F$ be taken as the pole and the bank $FX$ as initial line. Let at time $t$, the position of the boat be at $P(r, \theta)$ referred to $F$ as pole. The boat at $P$ will have 2 velocities, $u$ along $PF$ and $v$ parallel to $FX$.




At $P$, the radial velocity

$$\frac{dr}{dt} = v\cos\theta - u \qquad \text{...(i)}$$

and the transverse velocity

$$r\frac{d\theta}{dt} = -v\sin\theta \qquad \text{...(ii)}$$

Eq. (i)/ Eq. (ii) $\Rightarrow$ $$\frac{dr}{r\,d\theta} = \frac{v\cos\theta - u}{-v\sin\theta}$$

or $$\frac{dr}{r} = \left(\frac{u}{v}\operatorname{cosec}\theta - \cot\theta\right)d\theta$$

Integrating, $$\log r = \frac{u}{v}\log\tan\left(\frac{\theta}{2}\right) - \log\sin\theta + \log C$$

or $$\log r + \log\sin\theta - \log C = \frac{u}{v}\log\tan\left(\frac{\theta}{2}\right) \quad \text{or} \quad [(r\sin\theta)/C] = \left[\tan\left(\frac{\theta}{2}\right)\right]^{u/v},$$

which is the required equation.

If $v = u$, the above equation reduces to $r\sin\theta = C\tan\left(\frac{\theta}{2}\right)$

or $$\frac{C}{r} = \frac{\sin\theta}{\tan\left(\frac{\theta}{2}\right)} = \frac{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)/\cos\left(\frac{\theta}{2}\right)} = 2\cos^2\left(\frac{\theta}{2}\right) = 1 + \cos\theta$$

or $\frac{C}{r} = 1 + \cos\theta$, which is a parabola whose focus is the pole $F$.

**Example 13.** *A point moves in a plane curve, so that its tangential and normal accelerations are equal and the angular velocity of the tangent is constant. Find the curve.*

**Solution** Given $$v\frac{dv}{ds}=\frac{v^2}{\rho} \quad \text{...(i)}$$

and $$\frac{d\psi}{dt}=\omega = \text{constant} \quad \text{...(ii)}$$

From Eq. (i), $$\frac{dv}{ds}=\frac{v}{\rho}=\frac{v}{(ds/d\psi)}$$

Integrating it we get, $$\log v=\psi+\log C$$

or $$\log(v/C)=\psi \quad \text{or} \quad v=Ce^{\psi}$$

or $$\frac{ds}{dt}=Ce^{\psi} \quad \text{or} \quad \frac{ds}{d\psi}\cdot\frac{d\psi}{dt}=Ce^{\psi} \quad \text{or} \quad \frac{ds}{d\psi}\omega=Ce^{\psi} \quad \text{[From Eq. (ii)]}$$

Integrating, $$s=\frac{C}{\omega}e^{\psi}+k \quad \text{or} \quad s=Ae^{\psi}+B$$

where $A$ and $B$ are arbitrary constants.
This is the required intrinsic equation of the curve.

**Example 14.** *A particle is describing a plane curve. If the tangential and normal accelerations are each constant throughout the motion, prove that the angle* $\theta$*, through which the direction of motion turns in time* $t$*, is given by* $\theta = A\log(1+Bt)$

**Solution** Given that $$\frac{d^2s}{dt^2}=k \quad \text{... (i)}$$

and $$\frac{v^2}{\rho}=\lambda \quad \text{...(ii)}$$

where $k$ and $\lambda$ are constants.

Integrating Eq. (i), we get $$\frac{ds}{dt}=kt+C \quad \text{...(iii)}$$

From Eq. (ii), we get $$\frac{v^2}{\rho}=\lambda$$

or $$\frac{v^2}{ds/d\theta}=\lambda, \text{ where } v=\frac{ds}{dt}$$

or $$\frac{(ds/dt)^2}{ds/d\theta}=\lambda \quad \text{or} \quad \frac{ds}{dt}\cdot\frac{d\theta}{dt}=\lambda \quad \text{or} \quad (kt+C)\frac{d\theta}{dt}=\lambda \quad \text{[From Eq. (iii)]}$$

or $$d\theta=\frac{\lambda}{(kt+C)}dt$$

Integrating, $$\theta=\frac{\lambda}{k}\log(kt+C)+\log\mu$$

Let $\theta=0$ when $t=0$, then $$0=\frac{\lambda}{k}\log C+\log\mu$$

$\therefore$ $$\theta=\frac{\lambda}{k}\log(kt+C)-\frac{\lambda}{k}\log C$$
$$=\frac{\lambda}{k}\log\left(\frac{kt+C}{C}\right)$$

or $$\theta=\frac{\lambda}{k}\log\left[1+\frac{kt}{C}\right] \quad \text{or} \quad \theta=A\log[1+Bt]$$

where $$A=\frac{\lambda}{k} \quad \text{and} \quad B=\frac{k}{C}$$

**Example 15.** *A particle, projected with a velocity u is acted upon by a force which produces a constant acceleration a in the plane of the motion inclined at a constant angle α with the direction of motion. Obtain the intrinsic equation of the curve described, and show that the particle will be moving in the opposite direction to that of projection at time* $\dfrac{u}{a\cos\alpha}(e^{\pi\cot\alpha}-1)$

**Solution** Since the acceleration $a$ is inclined at an angle $\alpha$ with the direction of motion, hence, the equation of motion along tangential and normal directions are given by

$$v\frac{dv}{ds}=a\cos\alpha \quad \text{...(i)}$$

and
$$\frac{v^2}{\rho}=a\sin\alpha \quad \text{...(ii)}$$

Integrating Eq. (i), we get $\dfrac{1}{2}v^2=as\cos\alpha+C$

Let $s=0$ initially *i.e.*, when $v=u$.

Then $\dfrac{1}{2}u^2=0+C$ or $C=\dfrac{1}{2}u^2$

$$\therefore \quad v^2=2as\cos\alpha+u^2 \quad \text{...(iii)}$$

Substituting this value of $v^2$ in Eq. (ii), we get

$$2as\cos\alpha+u^2=a\rho\sin\alpha=a\cdot\frac{ds}{d\psi}\sin\alpha \quad (\because \rho=ds/d\psi)$$

$$=\frac{ads}{2as\cos\alpha+u^2}=\frac{d\psi}{\sin\alpha}$$

Multiplying both sides by 2 cos α then,we get

$$\frac{2a\cos\alpha\, ds}{2a\cos\alpha+u^2}=2\cot\alpha\, d\psi$$

Integrating, $\log(2as\cos\alpha+u^2)=2\psi\cot\alpha+\log A$

Let $\psi=0$ when $s=0$, the $\log u^2=\log A$

$$\therefore \quad \log(2as\cos\alpha+u^2)=2\psi\cot\alpha+\log u^2$$

$$\frac{u^2+2as\cos\alpha}{u^2}=e^{2\psi\cot\alpha} \quad \text{... (iv)}$$

$$[(2a\cos\alpha)/u^2]s=e^{2\psi\cot\alpha}-1$$

$$s=[u^2/(2a\cos\alpha)](e^{2\psi\cot\alpha}-1)$$

is the required intrinsic equation of the path

Again from Eq. (i), we get $d^2s/dt^2=a\cos\alpha$

Integrating, $\dfrac{ds}{dt}=at\cos\alpha+k$

when $t=0$, $\dfrac{ds}{dt}=u$, $\therefore$ $u=0+k$

Hence
$$\frac{ds}{dt}=at\cos\alpha+u \quad \text{...(v)}$$

Also from Eq. (iii), we have $v=\sqrt{(2as\cos\alpha+u^2)}$

$$\frac{ds}{dt}=\sqrt{u^2e^{2\psi\cot\alpha}} \quad \text{[From Eq. (iv)]}$$

$$\frac{ds}{dt}=ue^{\psi\cot\alpha}$$

$\therefore$ From Eq. (v), we get

$$ue^{\psi \cot \alpha} = at \cos \alpha + u$$

$$t = u\,(e^{\psi \cot \alpha} - 1)/a \cos \alpha \qquad \ldots\text{(vi)}$$

When the particle is moving in a direction opposite to that of projection (*i.e.*, opposite to $\psi = 0$).

We have $\psi = \pi$ and hence putting $\psi = \pi$ in Eq. (vi), we have the required time

$$t = u\,(e^{\pi \cot \alpha} - 1)/a \cos \alpha$$

**Example 16.** *A bungee jumper is attached to one end of a long elastic rope. The other end of the elastic rope is fixed to a high bridge. The jumper steps off the bridge and falls from rest towards the river below. He does not hit the water. The mass of the jumper is m. The unstretched length of the rope is L. The rope has a force constant (Force to produce 1m extension) of k and the gravitational field strength is g.*

*You may assume that the jumper can be regarded as a point mass m attached to the end of the rope. The mass of the rope is negligible compared to m, the rope obeys Hook's law, air resistance can be ignored throughout the fall of the jumper.*

*Obtain expression for the following*

*(a) The distance y dropped by the jumper before coming instantaneously to rest for the first time.*

*(b) The maximum speed v attained by the jumper during this drop.*

*(c) The time t taken during the drop before coming to rest for the first time.*

**Solution** (a) The jumper comes to rest when lost gravitational potential energy

= stored strain energy

$$mgy = \frac{1}{2} k (y - L)^2$$

$$ky^2 - 2y\,(kL + mg) + kL^2 = 0$$

This is solved as a quadratic $y = \dfrac{2\,(kL + mg) \pm \sqrt{4\,(kL + mg)^2 - 4k^2 L^2}}{2k}$

$$= \frac{kL + mg \pm \sqrt{2\,mg\,kL + m^2 g^2}}{k}$$

**Need positive root** Lower position of rest (other root after initial rise).

(b) The maximum speed is attained when the acceleration is zero and force balance.

*i.e.*, when $mg = kx$

Also kinetic energy = lost potential energy − strain energy within rope

$$\frac{1}{2} mv^2 = mg\,(L + x) - \frac{1}{2} kx^2, \text{where } x = \frac{mg}{k}$$

$$v^2 = 2g\left(L + \frac{mg}{k}\right) - \frac{mg^2}{k}$$

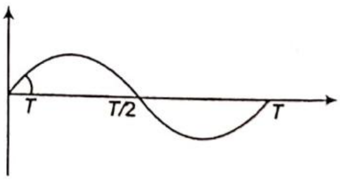
$$v = \sqrt{2\,gL + \frac{mg^2}{k}}$$

(c) Time to come to rest = Time in free fall

+ Time in SHM of rope to stop stretching

Length of free fall $= L = \dfrac{1}{2} g t_f^2$

So, $\qquad t_f = \sqrt{\dfrac{2\,L}{g}}$




We represent a full SHM cycle by the diagram.

The jumper enters the SHM with free fall velocity $= gt_f = \sqrt{2gL} = v_0$

Period of SHM $T = 2\pi\sqrt{\dfrac{m}{k}}$

The jumper enters the SHM at time $t$ given by $t = \dfrac{1}{\omega}\sin^{-1}\dfrac{v_0}{v} = \dfrac{1}{\omega}\sin^{-1}\dfrac{\sqrt{2gL}}{v}$

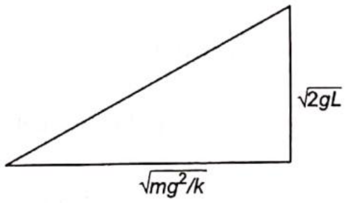
Jumper comes to rest at one half cycle of the SHM.

Total time given by $= t_f + \left(\dfrac{T}{2} - \tau\right)$

$$= \sqrt{\frac{2L}{g}} + \pi\sqrt{\frac{m}{k}} - \frac{1}{\omega}\sin^{-1}\frac{\sqrt{2gL}}{v} = \sqrt{\frac{2L}{g}} + \pi\sqrt{\frac{m}{k}} - \frac{1}{\omega}\sin^{-1}\frac{\sqrt{2gL}}{\sqrt{2gL + mg^2/k}}$$

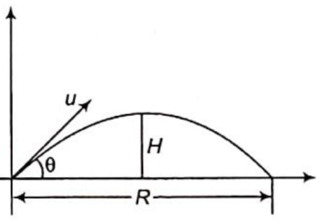
$$= \sqrt{\frac{2L}{g}} + \sqrt{\frac{m}{k}}\left\{\pi - \sin^{-1}\frac{\sqrt{2gL}}{\sqrt{2gL + mg^2/k}}\right\}$$

This is the same as $= \sqrt{\dfrac{2L}{g}} + \sqrt{\dfrac{m}{k}}\left\{\dfrac{\pi}{2} + \cos^{-1}\dfrac{\sqrt{2gL}}{\sqrt{2gL + mg^2/k}}\right\}$




$$= \sqrt{\frac{2L}{g}} + \sqrt{\frac{m}{k}}\tan^{-1}\left\{-\sqrt{\frac{2kL}{mg}}\right\}$$

## Projectile Motion




(a) Time of flight $T = \dfrac{2u\sin\theta}{g}$

(b) Horizontal range $R = \dfrac{u^2\sin 2\theta}{g}$

(c) Maximum height $H = \dfrac{u^2\sin^2\theta}{2g}$

(d) Maximum horizontal range $R_{\max} = \dfrac{u^2}{g}$ for $\theta = 45°$

(e) Equation of trajectory for projectile $y = u\sin\theta\left(\dfrac{x}{u\cos\theta}\right) - \dfrac{g}{2}\left(\dfrac{x}{u\cos\theta}\right)^2$

$$\therefore \qquad y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}$$

(f) Velocity of projectile at any instant of time $v = \sqrt{u^2 + g^2t^2 - 2u\sin\theta\, gt}$

(g) Angle made by $v$ with horizontal at time $t$, $\tan\alpha = \dfrac{u\sin\theta - gt}{u\cos\theta}$

$$\alpha = \tan^{-1}\left[\frac{u\sin\theta - gt}{u\cos\theta}\right]$$

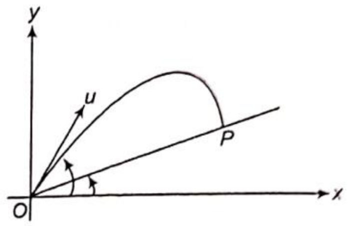
(h) Velocity at any height $h$, $\quad v = \sqrt{u^2 - 2gh}$

$$\tan\alpha = \frac{\sqrt{u^2\sin^2\theta - 2gh}}{u\cos\theta}$$

$$\alpha = \tan^{-1}\left[\frac{\sqrt{u^2\sin^2\theta - 2gh}}{u\cos\theta}\right]$$

# Projectile Motion Along an Inclined Plane

## Motion Up an Inclined Plane




Component of $u$ along plane

$= u\cos(\alpha - \beta)$

Component of $u$ perpendicular to plane

$= u\sin(\alpha - \beta)$

Component of $g$ along plane

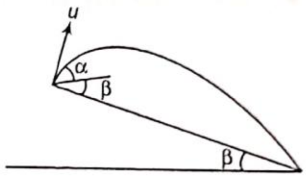
$= g\sin\beta$

Component of $g$ perpendicular to plane

$= g\cos\beta$

1. Time of flight $T = \dfrac{2u\sin(\alpha - \beta)}{g\cos\beta}$
2. Range up the inclined plane $R = \dfrac{2u^2\sin(\alpha - \beta)\cos\alpha}{g\cos^2\beta}$
3. Maximum range up the incline $R_{\max} = \dfrac{u^2}{g(1 + \sin\beta)}$

## Motion Down an Inclined Plane




1. Time of flight $T = \dfrac{2u\sin(\alpha + \beta)}{g\cos\beta}$
2. Range down an inclined plane $R = \dfrac{2u^2\sin(\alpha + \beta)\cos\alpha}{g\cos^2\beta}$
3. Maximum range down an inclined plane $R_{\max} = \dfrac{u^2}{g(1 - \sin\beta)}$

## Illustrative Solved Examples

**Example 17.** *Find direction of projection for maximum range for a given velocity of projection*

(*a*) *up an inclined plane,*

(*b*) *down an inclined plane.*

**Solution**

(a) Range of up an inclined plane is

$$R = \frac{2u^2\sin(\alpha - \beta)\cos\alpha}{g\cos^2\beta}$$

where $u$ = velocity of projection

$\alpha$ = angle of projection w.r.t. horizontal

$\beta$ = angle of inclination of plane

$$R = \frac{u^2}{g\cos^2\beta}[2\sin(\alpha - \beta)\cos\alpha] \qquad [2\sin C\cos D = \sin(C + D) + \sin(C - D)]$$

$$= \frac{u^2}{g\cos^2\beta}[\sin(2\alpha - \beta) - \sin\beta]$$

$R$ is maximum when $\sin(2\alpha - \beta)$ is maximum.

$$\sin(2\alpha - \beta) = 1 \quad \Rightarrow \quad 2\alpha - \beta = 90°$$

$$2\alpha = 90° + \beta$$

$$\alpha = \frac{90° + \beta}{2}$$

(b) Range down an inclined plane

$$R = \frac{2u^2 \sin(\alpha + \beta)\cos\alpha}{g\cos^2\beta}$$

$$R = \frac{u^2 [2\sin(\alpha + \beta)\cos\alpha]}{g\cos^2\beta}$$

$$R = \frac{u^2}{g\cos^2\beta}[\sin(2\alpha + \beta) + \sin\beta]$$

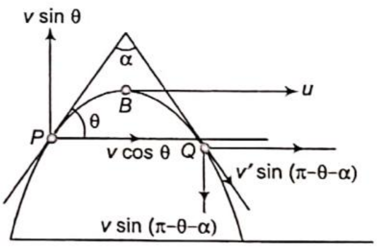
$R$ is maximum when $\sin(2\alpha + \beta)$ is maximum $\sin(2\alpha + \beta) = 1$

$$2\alpha + \beta = 90°$$

$$\alpha = \frac{90° - \beta}{2}$$

**Example 18.** *If $\alpha$ be the angle between tangents at the extremities of any arc of a parabolic path, $v$ and $v'$ the velocities at these extremities and $u$ the velocity at the vertex of the path, show that the time of describing the arc is $\frac{vv'\sin\alpha}{ug}$.*

**Solution.** Let $PQ$ be the arc of the parabolic path under consideration. Let $\theta$ be the angle which direction of motion of the particle at $P$ makes with the horizontal, then the direction of motion at $Q$ makes an angle $(\pi - \theta - \alpha)$ with the horizontal.




Horizontal component of velocity remains constant throughout the motion.

$\therefore$
$$v\cos\theta = u = v'\cos(\pi - \theta - \alpha)$$

$$v\cos\theta = u = -v'\cos(\theta + \alpha) \qquad \text{...(i)}$$

Also for the vertical component of velocity for the motion from $P$ to $Q$, we have

$$-v'\sin(\pi - \theta - \alpha) = v\sin\theta - gt$$

or
$$gt = v\sin\theta + v'\sin(\theta + \alpha) = (v\sin\theta) \times 1 + v'\sin(\theta + \alpha) \times 1$$

$$= v\sin\theta\left[\frac{-v'\cos(\theta + \alpha)}{u}\right] + v'\sin(\theta + \alpha)\left[\frac{v\cos\theta}{u}\right] \qquad \text{[From Eq. (i)]}$$

or
$$t = \frac{vv'}{ug}[\sin(\theta + \alpha)\cos\theta - \cos(\theta + \alpha)\sin\theta]$$

or
$$t = \frac{vv'}{ug}\sin[(\theta + \alpha) - \theta]$$

or
$$t = \frac{vv'}{ug}\sin\alpha.$$

**Example 19.** *A particle is projected under gravity with a velocity $u$ in a direction making an angle $\alpha$ with the horizon. Show that the amount of deviation $D$ in direction of motion of the particle is given by :*

$$\tan D = (gt\cos\alpha)/(u - gt\sin\alpha)$$

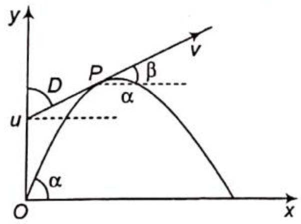
**Solution** After time $t$, let the particle be at $P$. Let the direction of motion of the particle at $P$ be inclined at an angle $\beta$ to the horizon and $v$ be the velocity there.

Then deviation $D$ in the direction of motion of the particle during this time $t$ = angle between the tangents at $O$ and $P = \alpha - \beta$ ...(i)

Horizontal component of velocity remains constant through the motion, so

$$v\cos\beta = u\cos\alpha \qquad ...(ii)$$

Also for the vertical motion from $O$ to $P$, we have




$$v\sin\beta = u\sin\alpha - gt \qquad ...(iii)$$

$$\text{Eq. (iii)/ Eq. (ii)} \Rightarrow \tan\beta = \frac{u\sin\alpha - gt}{u\cos\alpha} \qquad ...(iv)$$

or
$$\tan\beta = \tan\alpha - \frac{gt}{u\cos\alpha}$$

or
$$\tan\alpha - \tan\beta = \frac{gt}{u\cos\alpha} \qquad ...(v)$$

Now,
$$\tan D = \tan(\alpha - \beta) \qquad \text{[From Eq. (i)]}$$

$$= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} = \frac{gt/u\cos\alpha}{1 + \tan\alpha\,[(u\sin\alpha - gt)/u\cos\alpha]} = \frac{gt}{u\cos\alpha + u\sin\alpha\tan\alpha - gt\tan\alpha}$$

$$= \frac{gt\cos\alpha}{u - gt\sin\alpha}$$

**Example 20.** *A gun is firing from the sea level out to sea. It is then mounted on a battery h feet higher up and fixed at the same elevation α. Show that the range is increased by*

$$\frac{1}{2}\left[\left(1 + \frac{2gh}{u^2\sin^2\alpha}\right)^{\frac{1}{2}} - 1\right]$$ *of itself, u being the velocity of projection.*

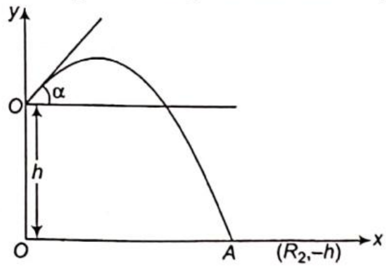
**Solution** Let $R_1$ be the range when the gun is firing from the sea level. Then we have

$$R_1 = (2u^2\sin\alpha\cos\alpha)/g \qquad ...(i)$$

Now, let the gun be mounted on the battery at a height $h$ feet above the sea level.

Then referring to $O$, the point of projection, as origin and $Ox$ and $Oy$ as co-ordinate axes, the co-ordinates of $A$, where the shot strikes the water are $(R_2, -h)$, where $R_2$ is the range in this case *i.e.*, $O'A = R_2$.

The equation of the path of this shot is




$$y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}$$

$$A\,(R_2, -h) \text{ lies on it so } -h = R_2\tan\alpha - \frac{gR_2^2}{2u^2\cos^2\alpha}$$

or
$$gR_2^2 - 2u^2\sin\alpha\cos\alpha\, R_2 - 2u^2 h\cos^2\alpha = 0$$

or
$$gR_2^2 - gR_1R_2 - 2u^2 h\cos^2\alpha = 0, \qquad \text{[From Eq.(i)]}$$

or
$$gR_2^2 - gR_1R_2 - 2u^2 h\left(\frac{gR_1}{2u^2\sin\alpha}\right)^2 = 0 \qquad \text{[From Eq.(i)]}$$

or
$$R_2^2 - R_1R_2 - \frac{hgR_1^2}{2u^2\sin^2\alpha} = 0$$

or $$\left(R_2 - \frac{R_1}{2}\right)^2 = \frac{R_1^2}{4} + \frac{hgR_1^2}{2u^2 \sin^2 \alpha} = \frac{R_1^2}{4}\left[1 + \frac{2gh}{u^2 \sin^2 \alpha}\right]$$

or $$R_2 - \frac{R_1}{2} = \frac{R_1}{4}\left[1 + \frac{2gh}{u^2 \sin^2 \alpha}\right]^{1/2} - \frac{R_1}{2}$$

Subtracting $\frac{1}{2}R_1$ from both sides

or $$R_2 - R_1 = \frac{R_1}{2}\left[\left(1 + \frac{2gh}{u^2 \sin^2 \alpha}\right) - 1\right]$$

**Example 21.** *Particles are projected from the same point in a vertical plane with velocity $\sqrt{2kg}$; prove that the locus of the vertices of their paths is the ellipse $x^2 + 4y(y - k) = 0$*

**Solution** If $\alpha$ be the angle of projection and $(x_1, y_1)$ be the co-ordinates of the vertex of one of the trajectories,

then $$x_1 = \frac{u^2 \sin \alpha \cos \alpha}{g} = \frac{2gk \sin \alpha \cos \alpha}{g} \qquad (\because u = \sqrt{2gk})$$

$$= 2k \sin \alpha \cos \alpha$$

and $$y_1 = \frac{u^2 \sin^2 \alpha}{2g} = \frac{2gk \sin^2 \alpha}{2g} = k \sin^2 \alpha$$

Eliminating $x$ from these we get

$$x_1^2 = 4k^2 \sin^2 \alpha \cos^2 \alpha = 4k^2 \left(\frac{y_1}{k}\right)\left(1 - \frac{y_1}{k}\right) = 4y_1 (k - y_1)$$

or $$x_1^2 + 4y_1(y_1 - k) = 0$$

Generalising, the required locus is $x^2 + 4y(y - k) = 0$

**Example 22.** *A particle is projected with velocity $u$ from a point on a plane inclined at angle $\alpha$ to the horizontal. If $r$ and $r'$ be the maximum ranges up and down the incline plane, prove that $\frac{1}{r} + \frac{1}{r'}$ is independent of inclination of the plane.*

**Solution** $r$ = maximum range up the plane $= \dfrac{u^2}{g(1 + \sin \alpha)}$

and $r'$ = maximum range down the plane $= \dfrac{u^2}{g(1 - \sin \alpha)}$

$$\therefore \quad \frac{1}{r} + \frac{1}{r'} = \frac{g(1 + \sin \alpha)}{u^2} + \frac{g(1 - \sin \alpha)}{u^2} = \frac{2g}{u^2}$$

Which is independent of $\alpha$.

**Example 23.** *Show that the greatest range up an inclined plane through the point of projection is equal to the distance through which a particle could fall freely during corresponding time of flight.*

**Solution** Let $u$ be the velocity of projection.

Let the plane be inclined at an angle $\beta$ to the horizontal.

For maximum range up the plane, the angle of projection $\alpha = \frac{1}{4}\pi + \frac{1}{2}\beta$

Also time of flight up the inclined plane $= \dfrac{2u \sin(\alpha - \beta)}{g \cos \beta} = \dfrac{2u \sin\left(\frac{1}{4}\pi + \frac{1}{2}\beta - \beta\right)}{g \cos \beta}$

$$= \frac{2u\sin\left(\frac{1}{4}\pi - \frac{1}{2}\beta\right)}{g\cos\beta} = \frac{2u\left[\sin\frac{1}{4}\pi\cos\frac{1}{2}\beta - \cos\frac{1}{4}\pi\sin\frac{1}{2}\beta\right]}{g\cos\beta}$$

$$= \frac{2u\left[\cos\frac{1}{2}\beta - \sin\frac{1}{2}\beta\right]}{\sqrt{2g\cos\beta}} = \frac{u\sqrt{2}}{g\left(\cos\frac{1}{2}\beta + \sin\frac{1}{2}\beta\right)} = T \qquad \text{(say)}$$

$\therefore$ The distance fallen by a particle in this time

$$T = \frac{1}{2}gT^2$$

$$= \frac{1}{2}g\left[\frac{2u^2}{g^2\left(\cos\frac{1}{2}\beta + \sin\frac{1}{2}\beta\right)^2}\right] = \frac{u^2}{g\left(\cos\frac{1}{2}\beta + \sin\frac{1}{2}\beta\right)^2}$$

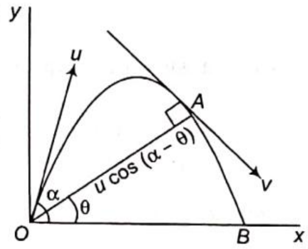
$\frac{u^2}{g(1+\sin\beta)}$ maximum range up the plane.

**Example 24.** *A particle is projected from O at an elevation $\alpha$. Prove that there are 2 positions A on its path at which the direction of velocity is perpendicular to OA. Prove that for real positions to exist, $\alpha$ is not less than $\cos^{-1}$ (1/3); also if in the two positions OA makes angles $\theta_1, \theta_2$ with the horizontal, prove that*

$$\theta_1 + \theta_2 = \alpha$$

**Solution** Let $u$ be the velocity of projection. Let the inclination of $OA$ to the horizontal be $\theta$. Then time taken by the particle in moving from $O$ to $A$

$$= 2[u\sin(\alpha - \theta)]/g\cos\theta \qquad \text{...(i)}$$




Considering the motion parallel to $OA$ we find that velocity of the particle parallel to $OA$ vanishes at $A$, since the direction of motion at $A$ is perpendicular to $OA$.

$\therefore$ From $\qquad v = u + at$, we find that

$$0 = u\cos(\alpha - \theta) - g\sin\theta \,.\, t$$

where $t =$ time taken from $O$ to $A$

or $\qquad t = [u\cos(\alpha - \theta)/g\sin\theta] \qquad \text{...(ii)}$

Eqs. (i) and (ii) give the time from $O$ to $A$

$$\therefore \qquad \frac{2u\sin(\alpha - \theta)}{g\cos\theta} = \frac{u\cos(\alpha - \theta)}{g\sin\theta}$$

or $\qquad 2\sin\theta\sin(\alpha - \theta) = \cos\theta\cos(\alpha - \theta)$

or $\qquad 2\sin\theta[\sin\alpha\cos\theta - \cos\alpha\sin\theta] = \cos\theta[\cos\alpha\cos\theta + \sin\alpha\sin\theta]$

or $\qquad 2\sin\alpha\sin\theta\cos\theta - 2\cos\alpha\sin^2\theta - \cos^2\theta\cos\alpha - \sin\alpha\sin\theta\cos\theta = 0$

or $\qquad 2\sin^2\theta\cos\alpha - \sin\alpha\sin\theta\cos\theta + \cos^2\theta\cos\alpha = 0$

or $\qquad 2\tan^2\theta\cos\alpha - \sin\alpha\tan\theta + \cos\alpha = 0 \qquad \text{...(iii)}$

This is a quadratic equation in $\tan\theta$, hence gives two values of $\tan\theta$ and corresponding to each value of $\tan\theta$ *i.e.*, $\theta$ there will be one position of $A$.

Let the 2 roots of Eq. (iii) be $\tan\theta_1$ and $\tan\theta_2$.

Then $$\tan\theta_1 + \tan\theta_2 = \frac{\sin\alpha}{2\cos\alpha} = \frac{1}{2}\tan\alpha$$

and $$\tan\theta_1 \tan\theta_2 = \frac{\cos\alpha}{2\cos\alpha} = \frac{1}{2}$$

$\therefore$ $$\tan(\theta_1 + \theta_2) = \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2} = \frac{\frac{1}{2}\tan\alpha}{1 - \frac{1}{2}} \text{ or } \theta_1 + \theta_2 = \alpha$$

Also from Eq. (iii), if $\tan\theta$ is real, then $B^2 - 4AC \geq 0$

or $$\sin^2\alpha - 4(2\cos\alpha)(\cos\alpha) \geq 0$$

or $$\sin^2\alpha - 8\cos^2\alpha \geq 0$$

or $$1 - \cos^2\alpha - 8\cos^2\alpha \geq 0$$

or $$1 - 9\cos^2\alpha \geq 0$$

or $$9\cos^2\alpha \leq 1$$

or $$\cos\alpha \leq 1/3$$

or $$\alpha \geq \cos^{-1}(1/3)$$

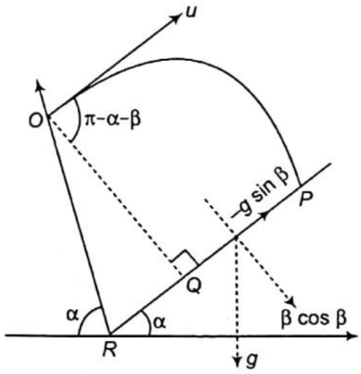
Hence, $\alpha$ cannot be less than $\cos^{-1}(1/3)$.

**Example 25.** *Two inclined planes intersect in a horizontal line, their inclinations to the horizontal being $\alpha$ and $\beta$; if a particle is projected at right angles to the former from a point in it so as to strike the other at right angles, the velocity of projection is*

$$\sin\beta\left[\frac{2ag}{\sin\alpha - \sin\beta\cos(\alpha+\beta)}\right]^{1/2}$$

*$a$ being the distance of the point of projection from the intersection of the planes.*

**Solution** Let $O$ be the point of projection and $u$ be the velocity of projection. Let the particle strike the other plane at $P$ at right angles. From $O$ draw $OQ$ perpendicular to the inclined plane $PR$




$$\angle ORQ = \text{angle between the planes}$$
$$= \pi - \beta - \alpha$$

$\therefore$ Direction of $u$ is at right angles to $OR$ and $OQ$ is at right angles to $PR$, therefore angle between direction of $u$ and $OQ$ is also $\pi - \alpha - \beta$.

$\therefore$ The component of $u$ along and perpendicular to $OQ$ are $u\cos(\pi - \alpha - \beta)$ and $u\sin(\pi - \alpha - \beta)$.

Also the components of acceleration along and perpendicular to $OQ$ are $g\cos\beta$ and $-g\sin\beta$.

Also $OR = a$ (given)

$\therefore$ From $\Delta\, ORQ$,

$$OQ = a\sin(\pi - \alpha - \beta) = a\sin(\alpha + \beta)$$

Considering the motion parallel to $OQ$ from $O$ to $P$, we have

$$OQ = u\cos(\pi - \alpha - \beta)t + \frac{1}{2}g\cos\beta\, t^2$$

where $t$ is the time taken in moving from $O$ to $P$

or $$a\sin(\alpha + \beta) = -u\cos(\alpha + \beta)t + \frac{1}{2}g\cos\beta\, t^2 \quad \ldots\text{(i)}$$

Also as the particles strikes the plane $PR$ at right angles at $P$ so the component of velocity perpendicular to $OQ$ vanishes at $P$.

Hence, we have for motion perpendicular to $OQ$

$$0 = u\sin(\pi - \alpha - \beta) - g\sin\beta\, t$$

or
$$t = \frac{u\sin(\alpha+\beta)}{g\sin\beta}$$

Substituting this value in Eq. (i), we get

$$a\sin(\alpha+\beta) = \frac{-u^2\cos(\alpha+\beta)\sin(\alpha+\beta)}{g\sin\beta} + \frac{g\cos\beta\, u^2\sin^2(\alpha+\beta)}{2g^2\sin^2\beta}$$

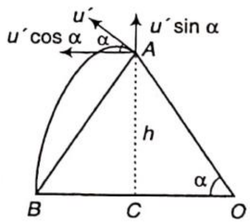
or
$$2ag\sin^2\beta = -2u^2\sin\beta\cos(\alpha+\beta) + u^2\cos\beta\sin(\alpha+\beta)$$
$$= u^2[\{\cos\beta\sin(\alpha+\beta) - \sin\beta\cos(\alpha+\beta)\} - \sin\beta\cos(\alpha+\beta)\}]$$
$$= u^2[\sin(\alpha+\beta-\beta) - \sin\beta\cos(\alpha+\beta)]$$

or
$$u = \left[\frac{2ag\sin^2\beta}{\sin\alpha - \sin\beta\cos(\alpha+\beta)}\right]^{1/2} = \sin\beta\left[\frac{2ag}{\sin\alpha - \sin\beta\cos(\alpha+\beta)}\right]^{1/2}$$

**Example 26.** *Two inclined planes of equal altitudes h and inclined at the same angle $\alpha$ to the horizon are placed back to back upon a horizontal plane. A ball is projected from the foot of one plane along its surface and in a direction making an angle $\beta$ with its line of intersection with the horizontal plane. After flying over the ridge it falls at the foot of other plane, show that the velocity of projection is*

$$\frac{1}{2}\sqrt{gh}\,\mathrm{cosec}\,\beta\sqrt{8+\mathrm{cosec}^2\alpha}$$

**Solution** $OAB$ is the normal cross-section of the given planes by a plane perpendicular to their common ridge. Let $O$ be the point of projection and $u$ be the velocity of projection.




Since the ball is projected from $O$ along the surface of the inclined plane containing $O$ and in a direction making an angle $\beta$ with its line of intersection with the horizontal plane, therefore the components of $u$ along $OA$ and horizontally are $u\sin\beta$ and $u\cos\beta$ respectively. The components $u\cos\beta$ remains constant throughout the motion and it would cause the particle to fall at some point $D$ instead of $B$, which is exactly opposite to $O$, such that $BD = u\cos\beta T$, where $T$ is the time of flight.

Here $D$ lies on the line of intersection of the second inclined plane with the horizontal plane.

Due to velocity component $u\sin\beta$ the ball will move first along the line $OA$ then leave the plane at $A$ and finally describe the parabolic path $AB$.

Hence we have two displacements of the ball, one due to velocity component $u\cos\beta$ and the other due to velocity component $u\sin\beta$. We shall consider them separately.

From $\Delta\, AOC$, $OA = h\,\mathrm{cosec}\,\alpha$ and $OC = h\cot\alpha = CB$

Let $u'$ be the velocity of the ball when it reaches $A$. Then, we have

$$u'^2 = u^2\sin^2\beta - 2g\sin\alpha\, OA$$
$$= u^2\sin^2\beta - 2g\sin\alpha\, h\,\mathrm{cosec}\,\alpha$$
$$= u^2\sin^2\beta - 2gh \quad \text{...(i)}$$

$\therefore$ At $A$, the horizontal and vertical components of velocity $u'$ are $u'\cos\alpha$ and $u'\sin\alpha$ respectively.

Let $t$ be the time taken in moving from $A$ to $B$, then considering the motion in horizontal and vertical directions, we get

$$(u'\cos\alpha)t = BC = h\cot\alpha \quad \text{...(ii)}$$

and $$-h = (u' \sin \alpha)t - \frac{1}{2}gt^2 \qquad \text{...(iii)}$$

From Eq. (ii) we get $$t = \frac{h}{u' \sin \alpha}$$

$\therefore$ From Eq. (iii) we get

$$-h = h - \frac{1}{2}g \frac{h^2}{u'^2 \sin^2 \alpha}$$

or $$(u')^2 = (gh \operatorname{cosec}^2\alpha)/4 \qquad \text{...(iv)}$$

Substituting this value of $u'^2$ in Eq. (i) we get

$$\frac{1}{4}gh \operatorname{cosec}^2 \alpha = u^2 \sin^2 \beta - 2gh$$

or $$u^2 \sin^2 \beta = \frac{1}{4}gh(8 + \operatorname{cosec}^2 \alpha)$$

or $$u = \frac{1}{2}\sqrt{gh} \operatorname{cosec} \beta \sqrt{8 + \operatorname{cosec}^2 \alpha}$$

As for $u \cos \beta$, it being the horizontal component, it will have no effect on the ball's vertical motion.

**Example 27.** *A gun fires a shell with a muzzle velocity u, show that the farthest horizontal distance at which an aeroplane at a height h can be hit is* $(u/g)\sqrt{u^2 - 2gh}$ *and the gun's elevation then is*

$$\tan^{-1}\left(\frac{u}{\sqrt{u^2 - 2gh}}\right)$$

**Solution** Let $\alpha$ be the angle of projection of the shell.

Let the point of projection of the shell be taken as origin and the horizontal line (lying in the plane of flight) and vertical line through the point of projection be taken as $x$ and $y$ axes respectively.

Then the equation of trajectory of the shell is

$$y = x \tan \alpha - \frac{gx^2}{2u^2 \cos^2 x} \qquad \text{...(i)}$$

If $R$ be the horizontal distance of the aeroplane from the point of projection, then the co-ordinates of the position of the aeroplane when it is hit by the shell are $(R, h)$

$(R, h)$ lies on Eq. (i),

so $$h = R \tan \alpha - \frac{gR^2 \sec^2 \alpha}{2u^2} \qquad \text{...(ii)}$$

If $R$ be maximum, then $\frac{dR}{dx} = 0$ and $\frac{d^2R}{dx^2}$ = negative

Differentiating both sides of Eq. (ii) with respect to $\alpha$, we have

$$0 = \left(R \sec^2 \alpha + \frac{dR}{d\alpha} \tan \alpha\right) - \frac{g}{2u^2}\left[2R\sec^2 \alpha \frac{dR}{d\alpha} + 2R^2 \sec^2 \alpha \tan \alpha\right]$$

or $$\frac{dR}{d\alpha}\left[\tan \alpha - \frac{gR \sec^2 \alpha}{u^2}\right] = \frac{gR^2 \sec^2 \alpha \tan \alpha}{u^2} - R \sec^2 \alpha$$

If $$\frac{dR}{d\alpha} = 0,$$

then $$\frac{gR^2\sec^2\alpha\tan\alpha}{u^2} - R\sec^2\alpha = 0$$

or $$\tan\alpha = u^2/gR \quad \because \sec^2\alpha \neq 0 \qquad \ldots(\text{iii})$$

Substituting this value of $\tan\alpha$ in Eq. (ii), we get

$$h = R\left(\frac{u^2}{gR}\right) - \frac{gR^2}{2u^2}\left(1 + \frac{u^4}{g^2R^2}\right) = \frac{u^2}{g} - \frac{gR^2}{2u^2} - \frac{u^2}{2g} = \frac{u^2}{2g} - \frac{gR^2}{2u^2}$$

or $$\frac{gR^2}{2u^2} = \frac{u^2}{2g} - h = \frac{u^2 - 2gh}{2g} \qquad \text{or} \qquad R^2 = \frac{u^2}{g^2}(u^2 - 2gh) \text{ or } R = \frac{u}{g}\sqrt{u^2 - 2gh}$$

Also from Eq. (ii), we get

$$\tan\alpha = \frac{u^2}{gR} = \frac{u^2}{u\sqrt{u^2 - 2gh}} \qquad \text{or} \qquad \alpha = \tan^{-1}\left[\frac{u}{\sqrt{u^2 - 2gh}}\right]$$

**Example 28.** *Three particles are projected from the same point in the same vertical plane with velocities $u, v, w$ at elevations $\alpha, \beta, \gamma$ respectively. Prove that the foci of their path will lie in a straight line if*

$$\frac{\sin 2(\beta - \gamma)}{u^2} + \frac{\sin 2(\gamma - \alpha)}{v^2} + \frac{\sin 2(\alpha - \beta)}{w^2} = 0$$

**Solution** Co-ordinates of foci of the parabolic paths of three particles are

$$\left(\frac{u^2 \sin 2\alpha}{2g}, \frac{-u^2\cos 2\alpha}{2g}\right); \left(\frac{v^2\sin 2\beta}{2g}, \frac{-v^2\cos 2\beta}{2g}\right) \text{ and } \left(\frac{w^2\sin 2\gamma}{2g}, \frac{-w^2\cos 2\gamma}{2g}\right) \text{ respectively.}$$

These foci will lie in the same line, if

$$\begin{vmatrix} (u^2\sin 2\alpha)/2g & -(u^2\cos 2\alpha)/2g & 1 \\ (v^2\sin 2\beta)/2g & -(v^2\cos 2\beta)/2g & 1 \\ (w^2\sin 2\gamma)/2g & -(w^2\cos 2\gamma)/2g & 1 \end{vmatrix} = 0$$

or $$\begin{vmatrix} u^2\sin 2\alpha & u^2\cos 2\alpha & 1 \\ v^2\sin 2\beta & v^2\cos 2\beta & 1 \\ w^2\sin 2\gamma & w^2\cos 2\gamma & 1 \end{vmatrix} = 0$$

or $$u^2\sin 2\alpha[v^2\cos 2\beta - w^2\cos 2\gamma] - u^2\cos 2\alpha[v^2\sin 2\beta - w^2\sin 2\gamma] + [v^2w^2\sin 2\beta\cos 2\gamma - v^2w^2\cos 2\beta\sin 2\gamma] = 0$$

or $$\Sigma u^2v^2[\sin 2\alpha\cos 2\beta - \cos 2\alpha\sin 2\beta] = 0$$

or $$u^2v^2\sin 2(\alpha - \beta) + v^2w^2\sin 2(\beta - \gamma) + w^2u^2\sin 2(\gamma - \alpha) = 0$$

or $$\frac{\sin 2(\alpha - \beta)}{w^2} + \frac{\sin 2(\beta - \gamma)}{u^2} + \frac{\sin 2(\gamma - \alpha)}{v^2} = 0$$

**Example 29.** *The tangents to a projectile's path form a $\Delta ABC$, the velocities are $v_1$ along $BC$, $v_2$ along $CA$ and $v_3$ along $AB$, show that*

$$\frac{BC}{v_1} + \frac{CA}{v_2} + \frac{AB}{v_3} = 0.$$

**Solution** Let the tangents $BC$, $CA$ and $AB$ which form $\Delta ABC$ be inclined at angles $\theta_1$, $\theta_2$ and $\theta_3$ respectively to the horizontal.

Then in $\Delta\ ABC$, we get

$\angle A =$ angle between $CA$ and $AB$ which are inclined at angles $\theta_2$ and $\theta_3$ to the horizontal *i.e.*,

$\angle A = \theta_2 - \theta_3$

Similarly $$\angle B = \theta_3 - \theta_1$$

$\therefore$ $$\angle C = \pi - (\angle A + \angle B) = \pi - (\theta_2 - \theta_1)$$

Now from $\Delta\ ABC$, we have

$$\frac{BC}{\sin A} = \frac{CA}{\sin B} = \frac{AB}{\sin C} \text{ or } \frac{BC}{\sin(\theta_1 - \theta_3)} = \frac{CA}{\sin(\theta_3 - \theta_1)} = \frac{AB}{\sin(\theta_1 - \theta_2)} = k \text{ (say)} \quad \ldots\text{(i)}$$

Also we know that horizontal component of velocity remains constant throughout the motion, so we get

$$v_1 \cos\theta_1 = v_2 \cos\theta_2 = v_3 \cos\theta_3 \text{ or } \frac{v_1}{\cos\theta_2 \cos\theta_3} = \frac{v_2}{\cos\theta_3 \cos\theta_1}$$

$$= \frac{v_3}{\cos\theta_1 \cos\theta_2} = \lambda \text{ (say)} \quad \ldots\text{(ii)}$$

$\therefore$ From Eq. (i) and Eq. (ii), we get

$$\frac{BC}{v_1} = \frac{k \sin(\theta_2 - \theta_3)}{\lambda \cos\theta_2 \cos\theta_3} = \frac{k(\sin\theta_2 \cos\theta_3 - \cos\theta_2 \sin\theta_3)}{\lambda \cos\theta_2 \cos\theta_3}$$

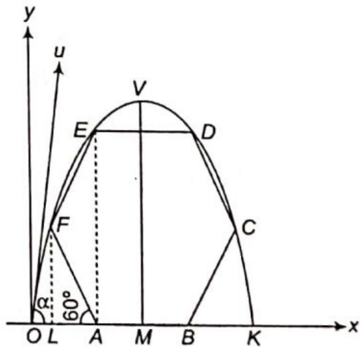
or $$\frac{BC}{v_1} = \frac{k}{\lambda}(\tan\theta_2 - \tan\theta_3)$$

Similarly $$\frac{CA}{v_2} = \frac{k}{\lambda}(\tan\theta_3 - \tan\theta_1)$$

and $$\frac{AB}{v_3} = \frac{k}{\lambda}(\tan\theta_1 - \tan\theta_2)$$

Adding these we get $\dfrac{BC}{v_1} + \dfrac{CA}{v_2} + \dfrac{AB}{v_3} = 0$

**Example 30.** *A particle is projected so as to graze the 4 upper corners of a regular hexagon whose side is c and which is placed vertically with one side on the table. Show that the range on the table is $c\sqrt{7}$ and that the square of the time of flight is $28c/g\sqrt{3}$.*




**Solution** Let the particle projected from the point $O$ with a velocity $u$ making an angle $\alpha$ with the horizontal.

Let the particle again strike the ground at $K$ after grazing the four upper points $F, E, D$ and $C$ of the hexagon $ABCDEF$. The velocity with which the particles strikes at $K$ is $u$ and the least velocity will be $u\cos\alpha$ at the highest point $V$ of the parabolic path.

Let $c = 2a$. From $F$ draw $FL$ perpendicular to $OA$. Let $OL = h$

$$AL = 2a\cos 60° = a$$

and $$FL = 2a \sin 60° = \sqrt{3}a$$

$\therefore$ $$EA = 2\ FL = 2a\sqrt{3}$$

The co-ordinates of $B$ and $E$ are $(h, a\sqrt{3})$ and $(h + a, 2a\sqrt{3})$ respectively.

Also the equation of parabolic path is

$$y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha} \quad \ldots\text{(i)}$$

$\because E(h+a, 2a\sqrt{3})$ and $F(h, a\sqrt{3})$ lies on Eq. (i)

$\therefore$ $$2a\sqrt{3} = (h+a)\tan\alpha - [g(h+a)^2 / 2u^2\cos^2\alpha] \quad \text{...(ii)}$$

and $$a\sqrt{3} = h\tan\alpha - [gh^2 / 2u^2\cos^2\alpha] \quad \text{...(iii)}$$

$$\text{Eq. (iii)} - \text{Eq. (ii)} \Rightarrow a\sqrt{3} = a\tan\alpha - \frac{g(2ha+a^2)}{2u^2\cos^2\alpha}$$

or $$\sqrt{3} = \tan\alpha - \frac{g(2h+a)}{2u^2\cos^2\alpha} \quad \text{...(iv)}$$

Also range on the horizontal plane $= OK$

$$OK = (2u^2\sin\alpha\cos\alpha)/g$$

or $$2\,OM = (2u^2\sin\alpha\cos\alpha)/g,$$

$M$ is the mid-point of $OK$.

or $$OL + LA + AM = (u^2\sin\alpha\cos\alpha)/g$$

or $$h + a + a = (u^2\sin\alpha\cos\alpha)/g$$

or $$u^2 = g(h+2a)/\sin\alpha\cos\alpha \quad \text{...(v)}$$

Substituting Eq. (v) in Eq. (iv) we get

$$\sqrt{3} = \tan\alpha - \frac{g(2h+a)(\sin\alpha)\cos\alpha}{2g(h+2a)\cos^2\alpha} = \tan\alpha - \frac{(2h+a)\tan\alpha}{(2h+4a)}$$

or $$\sqrt{3} = \frac{3a\tan\alpha}{2h+4a} \text{ or } \tan\alpha = \frac{2h+4a}{a\sqrt{3}} \quad \text{...(vi)}$$

$\therefore$ From Eq. (v), $2(h+2a) = (2u^2\sin\alpha\cos\alpha)/g$

or $$a\sqrt{3}\tan\alpha = (2u^2\sin\alpha\cos\alpha)/g \quad \text{[From Eq. (vi)]}$$

or $$u^2\cos^2\alpha = \frac{1}{2}a\sqrt{3}g \quad \text{...(vii)}$$

Also from Eq. (iii),

$$a\sqrt{3} = h\tan\alpha - [ga^2 / 2u^2\cos^2\alpha] = \frac{h(2h+4a)}{a\sqrt{3}} - \frac{gh^2}{a\sqrt{3}g} \quad \text{[From Eqs. (vi) and (vii)]}$$

or $$3a^2 = h^2 + 4ah \text{ or } h^2 + 4ah - 3a^2 = 0$$

or $$h = \frac{1}{2}[-4a \pm \sqrt{16a^2 + 12a^2}] = -2a + a\sqrt{7} \because h \text{ is +ve.}$$

or $$h + 2a = a\sqrt{7} \quad \text{...(viii)}$$

$\therefore$ From Eq. (vi) we get $\tan\alpha = \dfrac{2a\sqrt{7}}{a\sqrt{3}} = \dfrac{2\sqrt{7}}{\sqrt{3}}$

Hence, $\cos\alpha = \dfrac{\sqrt{3}}{\sqrt{31}}$ or $\sec^2\alpha = \dfrac{31}{3}$

$\therefore$ From Eq. (vii), $u^2 = \dfrac{a\sqrt{3}g}{2}\sec^2\alpha = \dfrac{31ag}{2\sqrt{3}}$

Hence, Range on the table, $OK = \dfrac{2u^2\sin\alpha\cos\alpha}{g} = \dfrac{2\times 31cg \times\sqrt{7}\times 2\sqrt{3}}{4\sqrt{3}\times g\sqrt{31}\times\sqrt{31}} = \dfrac{c}{\sqrt{7}}$

And square of the time of flight

$$= \left(\frac{2u\sin\alpha}{g}\right)^2 = \frac{4u^2\sin^2\alpha}{g^2} = \frac{4\times 31\times cg\times 28}{4\sqrt{3}g^2\times 31} = \frac{28c}{g\sqrt{3}}$$

**Example 31.** *A particle projected with velocity $\sqrt{2gh}$ so that its range on horizontal plane through the point of projection is $4r$. When it is at a distance $L$ from the point of projection it is moving at right angles to the original direction of motion, prove that*

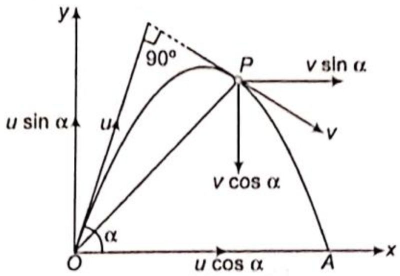
$$L^2r^2 = h^3(L-h)$$

**Solution** Let the particle be projected from $O$ with velocity $u = \sqrt{2gh}$ making angle $\alpha$ with the horizontal.

Let after time $t$ the particle reach $P$ with velocity $v$ moving at right angles to the original direction.

Now, $$v\sin\alpha = u\cos\alpha$$

or $$v = u\cot\alpha \quad \ldots(\text{i})$$




Considering the vertical motion from $O$ to $P$

we have $$-v\cos\alpha = u\sin\alpha - gt$$

or $$gt = u\sin\alpha + v\cos\alpha$$
$$= u\sin\alpha + [u\cos^2\alpha/\sin\alpha]$$

[From Eq. (i)]

or $$gt = \frac{u(\sin^2\alpha + \cos^2\alpha)}{\sin\alpha} = \frac{u}{\sin\alpha}$$

or $$t = \frac{u}{g\sin\alpha} \quad \ldots(\text{ii})$$

If co-ordinates of $P$ be $(x_1, y_1)$, then we get

$$x_1 = u\cos . t = u\cos\alpha\frac{u}{g\sin\alpha} \quad \text{[From Eq. (ii)]}$$
$$= (u^2\cot\alpha)/g = 2h\cot\alpha$$

or $$x_1 = 2h\cot\alpha$$

and $$y_1 = u\sin\alpha\, t - \frac{1}{2}gt^2 = u\sin\alpha\left(\frac{u}{g\sin\alpha}\right) - \frac{1}{2}g\left(\frac{u}{g\sin\alpha}\right)^2 \quad \text{[From Eq. (ii)]}$$
$$= \frac{u^2}{g} - \frac{u^2}{2g\sin^2\alpha} = \frac{u^2}{2g}(2-\operatorname{cosec}^2\alpha)$$

or $$y_1 = h(2-\operatorname{cosec}^2\alpha) \quad \ldots(\text{iii})$$

Now, $$OP = L \quad \text{(given)}$$

$\therefore$ $$L^2 = OP^2 = x_1^2 + y_1^2 = (2h\cot\alpha)^2 + h^2(2-\operatorname{cosec}^2\alpha)^2$$

or $$L^2 = 4h^2\cot\alpha + 4h^2 - 4h^2\operatorname{cosec}^2\alpha + h^2\operatorname{cosec}^4\alpha$$

or $$L^2 = h^2\operatorname{cosec}^4\alpha$$

or $$L = h\operatorname{cosec}^2\alpha \quad \ldots(\text{iv})$$

Also horizontal range = $4r$ (given)

or $$(2u^2\sin\alpha\cos\alpha)/g = 4r$$

or $$r = (u^2\sin\alpha\cos\alpha)/2g = (2gh\sin\alpha\cos\alpha)/2g$$

or $$r = h\sin\alpha\cos\alpha$$

$\therefore$ From Eq. (iv) and (v)

$$L^2r^2 = h^2\operatorname{cosec}^4\alpha \times h^2\sin^2\alpha\cos^2\alpha$$

or $$L^2r^2 = h^4\cot^2\alpha = h^3(h\cot^2\alpha) = h^3(h\operatorname{cosec}^2\alpha - h)$$

or $$L^2r^2 = h^3(L-h)$$

**Example 32.** *If $v_1, v_2, v_3$ are the velocities at three points $P, Q, R$ on the path of projectile where the inclinations to the horizontal of these velocities are $\alpha$, $\alpha - \beta$, $\alpha - 2\beta$ and if $t_1, t_2$ be the times of describing $PQ, QR$ respectively, prove that $v_3 t_1 = v_1 t_2$;*

$$\frac{1}{v_1} + \frac{1}{v_3} = \frac{2\cos\beta}{v_2}$$

**Solution** Since, the horizontal component of velocity remains constant,

$\therefore$ We have

$$v_1 \cos\alpha = v_2 \cos(\alpha - \beta) = v_3 \cos(\alpha - 2\beta) \quad \text{...(i)}$$

Also for the vertical motion of the particle from $P$ to $Q$, we have

$$v_2 \sin(\alpha - \beta) = v_1 \sin\alpha - gt_1 \quad \text{...(ii)}$$

and from $Q$ to $R$, we have

$$v_3 \sin(\alpha - 2\beta) = v_2 \sin(\alpha - \beta) - gt_2 \quad \text{...(iii)}$$

Now, $$\frac{1}{v_1} + \frac{1}{v_3} = \frac{\cos\alpha}{v_2\cos(\alpha - \beta)} + \frac{\cos(\alpha - 2\beta)}{v_2\cos(\alpha - \beta)} \quad \text{[From Eq. (i)]}$$

$$= \frac{\cos\alpha + \cos(\alpha - 2\beta)}{v_2\cos(\alpha - \beta)} = \frac{2\cos(\alpha - \beta)\cos\beta}{v_2\cos(\alpha - \beta)} = \frac{2\cos\beta}{v_2}$$

Also from Eq. (ii),

$$gt_1 = v_1 \sin\alpha - v_2 \sin(\alpha - \beta) \quad \text{...(iv)}$$

From (iii), $$gt_2 = v_2 \sin(\alpha - \beta) - v_3 \sin(\alpha - 2\beta) \quad \text{...(v)}$$

Multiplying Eq. (iv) by $v_3$ and Eq. (v) by $v_1$ and subtracting, we get

$$g(v_3 t_1 - t_2 v_1) = v_1 v_3 \sin\alpha - v_2 v_3 \sin(\alpha - \beta) - v_2 v_1 \sin(\alpha - \beta) + v_3 v_1 \sin(\alpha - 2\beta) \; g(v_3 t_1 - t_2 v_1)$$

$$= v_1 v_3\left[\sin\alpha - \frac{v_2}{v_1}\sin(\alpha - \beta) - \frac{v_2}{v_3}\sin(\alpha - \beta) + \sin(\alpha - 2\beta)\right]$$

$$= v_1 v_3\left[\sin\alpha - v_2\left(\frac{1}{v_1} + \frac{1}{v_3}\right)\sin(\alpha - \beta) + \sin(\alpha - 2\beta)\right]$$

$$= v_1 v_3 [\sin\alpha - 2\cos\beta \sin(\alpha - \beta) + \sin(\alpha - 2\beta)]$$

$$= v_1 v_3 [\{\sin\alpha + \sin(\alpha - 2\beta)\} - 2\cos\beta \sin(\alpha - \beta)]$$

$$= v_1 v_3 [\{2\sin(\alpha - \beta)\cos\beta\} - 2\cos\beta \sin(\alpha - \beta)]$$

$$= v_1 \times v_3 \times 0 = 0$$

$\therefore$ $$v_3 t_1 - v_1 t_2 = 0$$

$\because$ $$g \neq 0$$

or $$v_3 t_1 = v_1 t_2$$

**Example 33.** *Prove that when a shot is projected from a gun at any angle of elevation, the shot as seen from the point of projection will appear to descend past vertical target with uniform velocity.*

**Solution** Let $u$ and $\alpha$ be the velocity and angle of projection respectively.

$RS$ is the given vertical target. Let $OS = d$ (say)

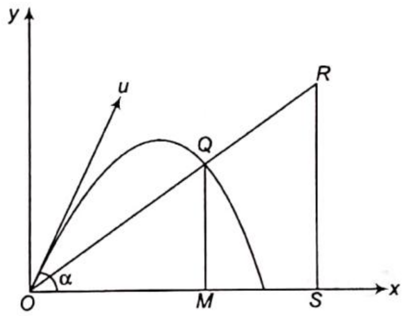
Let the particle be at $Q$ after time $t$. $QM$ is perpendicular from $Q$ to $x$-axis. Produce $OQ$ to meet the tangent in $R$.

$QM$ = Vertical distance travelled by the shot in time $t$

$$= u\sin\alpha \cdot t - \frac{1}{2}gt^2 \quad \text{...(i)}$$

and $OQ$ = horizontal distance moved by the shot in time $t$

$$= u\cos\alpha\, t$$




Also $\Delta\, OQM \sim \Delta\, ORS$

$\therefore$ $$\frac{RS}{OS} = \frac{QM}{OM}$$

or $$RS = OS \times \frac{QM}{OM}$$

or $$RS = \left(\frac{u \sin\alpha\, t - \frac{1}{2}gt^2}{u\cos\alpha\,.\,t}\right) \times d = \left(\tan\alpha - \frac{gt}{2u\cos\alpha}\right)d$$

If $(x, y)$ be the co-ordinates of $R$, then $x = OS = d$

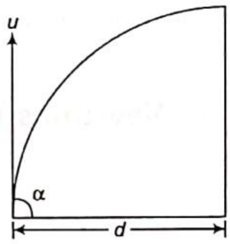
and $$y = RS = \left(\tan\alpha - \frac{gt}{2u\cos\alpha}\right)d$$

$\therefore$ $$\frac{dx}{dy} = 0 \text{ and } \frac{dy}{dt} = -\frac{gd}{2u\cos\alpha}$$

$\therefore$ The shot as seen from $O$ will appear to descend vertically downwards with uniform velocity.

**Example 34.** *A ball is projected with a velocity $u$ at an elevation $\alpha$ from point distance $d$ from a smooth vertical wall, it returns to the point of projection. Prove that $e = \dfrac{gd}{u^2 \sin 2\alpha - gd}$*

*where $e$ = coefficient of restitution. Hence find the maximum $d$ for which the ball can return to the point of projection.*




**Solution** The vertical force on the ball is only $mg$ throughout it motion because during impact it experiences only a horizontal force on the wall, we have

$$S_y = u_y\, t - (1/2)gt^2$$

Let $t$ = total time of flight

$$0 = u \sin\alpha\, t - \frac{1}{2}gt^2$$

or $$t = \frac{2u\sin\alpha}{g}$$

Due to impact with the wall, the normal component (*i.e.*, horizontal component) of velocity is reversed and becomes $e$ times the previous velocity.

Horizontal velocity before impact $= u\cos\alpha$

Hence, horizontal velocity after impact $= e\,u\cos\alpha$

Time taken to reach the wall $t_1 = \dfrac{d}{u\cos\alpha}$

Time taken to come back $t_2 = \dfrac{d}{e\,u\cos\alpha}$

Again $$t_1 + t_2 = t\ \frac{d}{u\cos\alpha} + \frac{d}{eu\cos\alpha} = \frac{2u\sin\alpha}{g}$$

$\Rightarrow$ $$1 + \frac{1}{e} = \frac{2u\sin\alpha}{g} \cdot \frac{u\cos\alpha}{d}$$

$\Rightarrow$ $$\frac{1}{e} = \frac{u^2\sin 2\alpha}{gd} - 1$$

$\Rightarrow$ $$e = \frac{gd}{u^2\sin 2\alpha - gd},$$ **Hence proved**

$$gd = \frac{u^2 \sin 2\alpha}{\left(1 + \frac{1}{e}\right)}$$

For maximum $d$, $\sin 2\alpha = 1$

$$d_{\max} = \frac{u^2}{g\left(1 + \frac{1}{e}\right)}$$

## Forces

(a) Force an interaction that cause an acceleration.

(b) Laws governing the Newtonian mechanics

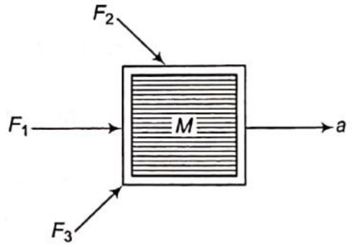
### (i) Newton's first law

If no force acts on a body we can always find a reference frame in which the body has no acceleration.

Or

If no force acts on a body, if the body is at rest, it will remain at rest, if the body is moving it will continue to do so.

### (ii) Newton's second law

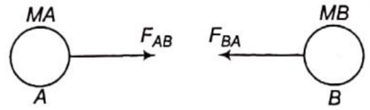



$$\Sigma \mathbf{F} = m\,\mathbf{a} \qquad m\,\mathbf{a} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \ldots$$

in scalar components

1. $\Sigma F_x = ma_x$ 2. $\Sigma F_y = ma_y$ 3. $\Sigma F_z = ma_z$

### (iii) Newton's third law

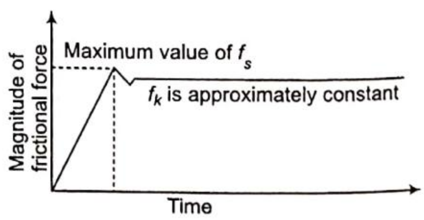



$$\mathbf{F}_{AB} = -\mathbf{F}_{BA}$$

Force of an action reaction pair are always equal and opposite and acts on different bodies.

## Friction

(a) **Friction is a contact force** It always opposes the relative motion between two surfaces.




(b) **Static friction**

$f_s \le \mu_s N$ (limiting friction)

where $N =$ Normal reaction

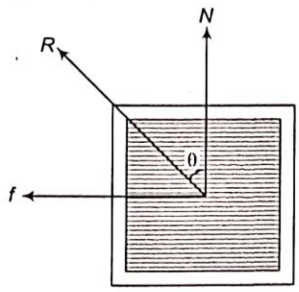
$f_s =$ Applied force $\le \mu_s N$

(c) **Kinetic friction**

$f_k = \mu_k N$

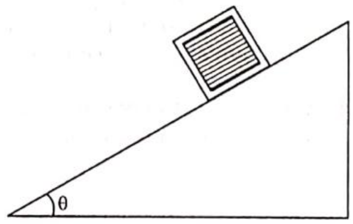
Applied force $\propto$ Time

(d) $\theta$ is called the angle of friction.




$$\mu = \tan\theta$$

(e) **Angle of repose** Minimum angle of inclination $\theta$ at which a body just start moving under its own weight.

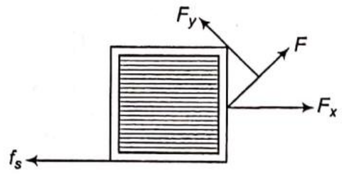



(f) **Properties of friction**

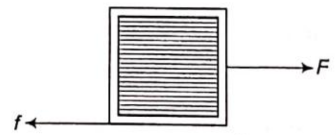
(i) If body is at rest

$f_s =$ component of $F$ anti-parallel to it.

(ii) $f_s \max = \mu_s N$




(iii) When body is moving, friction force is constant




$$f_k = \mu_k N.$$

(g) **Inertial and non-inertial frame of reference**

(i) Reference frame at rest $\mathbf{F}_{\text{rest}} = m\mathbf{a}_{\text{rest}}$

where $\mathbf{F}_{\text{rest}}$ and $\mathbf{a}_{\text{rest}}$ are force and acceleration of a body.

(ii) Reference frame moving with constant $v$ acceleration of frame = 0

$$\mathbf{a}_{\text{inertial}} = \mathbf{a}_{\text{rest}} - \mathbf{a}_{\text{frame}} = \mathbf{a}_{\text{rest}} - 0 = \mathbf{a}_{\text{rest}}$$

$$\mathbf{a}_{\text{inertial}} = \mathbf{a}_{\text{rest}}$$

(iii) Reference frame moves with constant acceleration.

Acceleration of frame $= \mathbf{a}_{\text{frame}}$

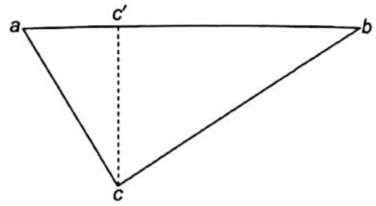
$a$ of mass w.r.t. frame $\mathbf{a}_{\text{rest}} = \mathbf{a}_{\text{inertial}} - \mathbf{a}_{\text{frame}} = \mathbf{a}_{\text{rest}} - \mathbf{a}_{\text{frame}}$

Let $\mathbf{F}_{\text{frame}}$ be force on mass in that frame $\mathbf{F}_{\text{frame}} = m\,\mathbf{a}_{\text{rest}} - m\,\mathbf{a}_{\text{frame}} = \mathbf{F}_{\text{rest}} + \mathbf{F}_{\text{pseudo}}$

where $\mathbf{F}_{\text{pseudo}} = -m\,\mathbf{a}_{\text{frame}}$

# Illustrative Solved Examples

**Example 35.** *Along which of the two trajectories, the horizontal line ac′ b or the broken line consisting of two straight segments (ac and cb), will the work performed by a force in displacing an object be greater, if the friction is the same for all three straight segments?*




**Solution** The work performed along $ac'$ is

$$A_1 = ac'\, mgk$$

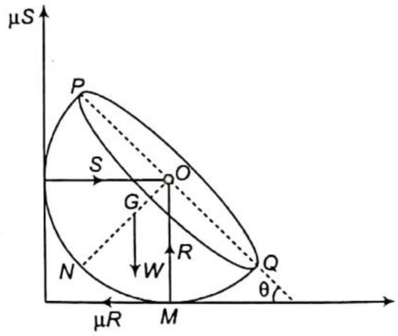
The work performed against the forces of friction on the inclined segment $ac$ is

$$A_2 = ac\, mgk\cos\alpha = \frac{ac'}{\cos\alpha} mgk\cos\alpha = ac'\, mgk$$

We see that the two quantities coincide, and so obviously, do the similar quantities for $c'b$ and $cb$. The change in the potential energy about $ac'b$ and $acb$ is zero. Thus, the work performed against the forces of friction along $ac'b$ and that performed against the forces of friction along $acb$ coincide.

**Example 36.** *A hemisphere rests in equilibrium on a rough ground and against an equally rough wall, if the equilibrium is limiting, the inclination of the base to the horizontal is*

$$\sin^{-1}\left[\frac{8}{3}\left(\frac{\mu+\mu^2}{1+\mu^2}\right)\right]$$




**Solution** Let $\theta$ be the required inclination of the base to the horizontal. Then the forces of friction $\mu R$ and $\mu S$ will be as marked in the figure, where $R$ and $S$ are the normal reactions at the points of contact $LM$ of the disc with the vertical and horizontal planes. Let $W$ be the weight of hemisphere acting at $G$, such that $OG = \left(\frac{3}{8}\right)a$, where $a$ is the radius and $O$ is the centre.

Resolving the forces horizontally and vertically, we get

$$S = \mu R \quad \text{...(i)}$$

$$R + \mu S = W \quad \text{...(ii)}$$

Eliminating $R$ between Eqs. (i) and (ii), we have

$$\frac{S}{\mu} + \mu S = W \Rightarrow S = \frac{\mu W}{(1+\mu^2)} \quad \text{...(iii)}$$

Also taking moments about the centre $O$, we have

$$\mu S \cdot OL + \mu R \cdot OM = W \cdot OG \sin\theta \quad \text{...(iv)}$$

$\therefore$ Substituting from Eqs.(i) and (ii) in Eq. (iv), we have

$$\frac{\mu^2 W}{(1+\mu^2)}.a + \frac{\mu W}{(1+\mu^2)}a = W.\frac{3a}{8}\sin\theta \quad \text{...(v)}$$

or

$$\sin\theta = \frac{8}{3}.\left(\frac{\mu+\mu^2}{1+\mu^2}\right)$$

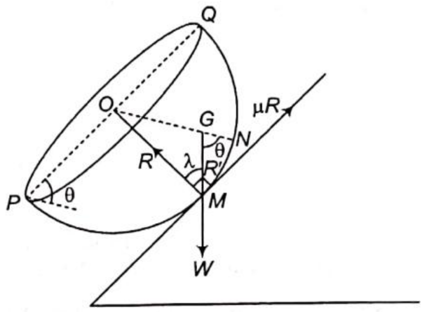
or

$$\theta = \sin^{-1}\left[\frac{8}{3}\left(\frac{\mu+\mu^2}{1+\mu^2}\right)\right]$$

**Hence proved.**

**Exampe 37.** *A hemispherical shell rests on a rough inclined plane, whose angle of friction is λ; then inclination of the plane base of the rim to the horizon cannot be greater than* $\sin^{-1}(2\sin\lambda)$.

**Solution** Let the base of the shell make the angle θ with the horizon in the position when it is on the point of slipping, then this inclination of the base is the greatest.




Then the forces are as marked in the figure. The shell is in equilibrium under the action of three forces *viz.*

(i) the normal reaction $R$ at the point of contact $M$,

(ii) the force of friction $\mu R$ and its weight $W$ acting through $G$,

(iii) the centre of gravity of the shell such that $OG = \frac{1}{2}a$, where $a$ is the radius of the shell.

Since, these three forces keep the shell in equilibrium, so $W$ will balance the resultant $R'$ of $R$ and $\mu R$ *i.e.*, $GM$ is a vertical line.

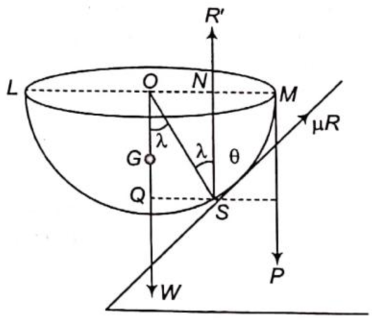
$\therefore$ $\angle MGN = \theta$ and $\angle OMG = \lambda$

In $\Delta OMG$, $\dfrac{OG}{\sin OMG} = \dfrac{OM}{\sin OGM}$ or $\dfrac{\frac{1}{2}a}{\sin\lambda} = \dfrac{a}{\sin(\pi - \theta)}$

or $2\sin\lambda = \sin\theta$ or $\theta = \sin^{-1}(2\sin\lambda)$ **Hence proved.**

**Example 38.** *A solid hemisphere of weight W rests with its curved surface in contact with a rough inclined plane. A weight P is placed at same point on the rim of the hemisphere to keep its plane surface horizontal. Then its coefficient of friction is* $P/\sqrt{\{W(W+2P)\}}$.

**Solution** Let $O$ be centre of the base of the hemisphere and $G$ be its centre of gravity, where $G$ is on the vertical radius of the hemisphere. The weight $W$ of the hemisphere acts at $G$ vertically downwards.




The weight $P$, attached at the point $M$ on the base of the hemisphere, acts vertically downwards, let $R$ be the normal reaction between the hemisphere and the inclined plane. Since the hemisphere is on the point of slipping downwards, so the force of friction $\mu R$ acts upwards along the inclined plane. Let $R'$ be the resultant reaction.

So, $\angle OSN = \lambda = \angle SOQ$ (By figure)

where $\lambda$ is the angle of friction.

$\therefore$ The hemisphere is in equilibrium under the action of three forces $W, P$ and $R'$, out of which two *viz.*, $W$ and $P$ are acting vertically downwards hence parallel, so the line of action of $R'$ is also vertical.

Taking moments of these forces about $S$, we get

$W \cdot SQ = P \cdot SX$ (See fig.) or $W \cdot OS\sin\lambda = P\cdot(OM - SQ) = P\cdot(OS - OS\sin\lambda)$

$\because$ $OM = OS =$ radius or $W\sin\lambda = P(1-\sin\lambda)$

or $(W+P)\sin\lambda = P$

$$\Rightarrow \quad \sin\lambda = \frac{P}{(P+W)} \qquad \ldots(i)$$

$$\therefore \quad \tan\lambda = \frac{1}{\cot\lambda}$$

$$= \frac{1}{\sqrt{\text{cosec}^2\lambda - 1}} = \frac{\sin\lambda}{\sqrt{(1-\sin^2\lambda)}}$$

$$\tan\lambda = \frac{P/(P+W)}{\sqrt{[1-\{P/(P+W)\}^2]}}$$

Putting the value of $\sin\lambda$ from Eq. (i)

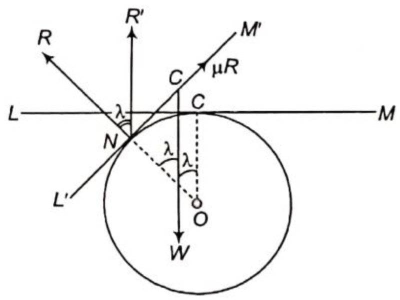
or
$$\mu = \frac{P}{\sqrt{[(P+W)^2 - P^2]}} = \frac{P}{\sqrt{[W^2 + 2PW]}}$$

where $\mu$ is the coefficient of friction.

or
$$\mu = \frac{P}{\sqrt{[W(W+2P)]}}.$$
**Hence proved.**

**Example 39.** *A uniform plank of length 2a and weight W rests with its middle point upon a rough horizontal cylinder whose axis is perpendicular to the plank. Show that the greatest weight that can be attached to one end of the plank, without sliding it off the cylinder is $\frac{b\lambda}{a-b\lambda}W$, where b is the radius of the cylinder and $\lambda$ the angle of friction.*

**Solution** Let the circle with centre $O$ be the vertical cross-section of the cylinder through $C$, the initial point of contact of the rod, when the greatest weight $P$ is attached at $L$, let the rod take the position $L'M'$ with $N$ as the new point of contact and its weight $W$ acting vertically downwards through $C'$, the mid-point of $L'M'$.




In this position the rod $L'M'$ is on the point of slipping downwards. Let $R'$ be the resultant reaction at $N$ *i.e.*, the resultant of the normal reaction $R$ and force of friction $\mu R$ at $N$.

The rod $L'M'$ is in equilibrium under the action of three force $R'$, $P$ and $W$ out of which two *viz.*, $W$ and $P$ are vertical *i.e.*, their lines of action are parallel, hence the third force $R'$ is also acting in the vertical direction.

Then $\angle CON =$ angle between $R$ and $R'$ = angle of friction $A$.

Also $C'N = \text{arc } CN = b\lambda$, since $OC = b$

$L'N = L'C' - C'N = a - b\lambda$ ($\because L'C' = a$)

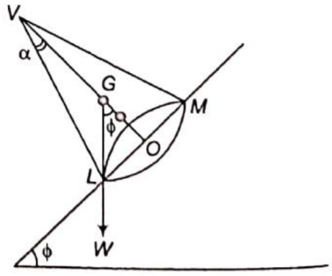
Now as $R'$ balances the resultant of $P$ and $W$, therefore,

$$P \cdot L'N = W \cdot C'N = a \quad \text{or} \quad P(a - b\lambda) = W(b\lambda)$$

or
$$P = \frac{b\lambda}{a - b\lambda} \cdot W$$
**Hence proved.**

**Example 40.** *A right cone is placed with its base on a rough inclined plane. If $\frac{1}{\sqrt{3}}$ be the coefficient of friction, find the angle of the cone when it is on the point of slipping and turning over.*

**Solution** When the cone is on the point of slipping, the inclination of the plane to the horizon is $\lambda$, given by $\mu = \tan\lambda$ ...(i)




Let the plane be inclined at an angle $\phi$ to the horizontal when cone is on the point of toppling over. In this case the vertical line through the centre of gravity $G$ of the cone will just fall within the base of the cone *i.e.*, will pass through $L$

$\therefore$ $\Delta\, LGO$, we have $\tan\phi = \frac{LO}{OG} = \frac{r}{4h}$

where, $r$ is the radius of the base and $h$ is the height of the cone.

$\therefore \qquad \tan\phi = \dfrac{4r}{h} = 4\tan\alpha, \qquad \left[\because \tan\alpha = \dfrac{LO}{VO} = \dfrac{r}{h}\right]$

Now, the cone will slide before it topples over if $\lambda < \phi$, then

$$\tan\lambda < \tan\phi \qquad \text{...(ii)}$$

But, if $\lambda$ and $\phi$ be the inclination of the plane to the horizontal, when the cone is on the point of slipping and turning over respectively, then

$$\tan\lambda = \mu = \frac{1}{\sqrt{3}} \text{ (Given)} \qquad \text{...(iii)}$$

[$\because$ From Eq. (i)]

$$\tan\phi = 4\tan\alpha \qquad \text{...(iv)}$$

[From Eqs. (i) and (ii)]

where $2\alpha$ is the vertical angle of the cone.

Now, if the cone is on the point of slipping as well as turning over at the same time, then

$$\lambda = \phi \text{ or } \tan\lambda = \tan\phi$$

or $\dfrac{1}{\sqrt{3}} = 4\tan\alpha$ [From Eqs. (iii) and (iv)]

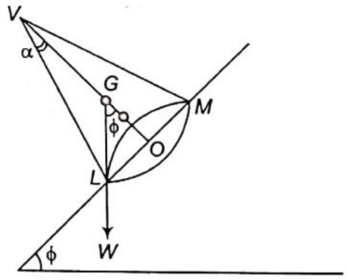
or $\tan\alpha = \dfrac{1}{4\sqrt{3}} \Rightarrow \alpha = \tan^{-1}\left(\dfrac{1}{4\sqrt{3}}\right)$

$\therefore$ Vertical angle of the cone $= 2\alpha = 2\tan^{-1}\left(\dfrac{1}{4\sqrt{3}}\right)$

**Example 41.** *A cone of given vertical angle $2\alpha$, rests on a rough plane which is inclined to the horizon. As the inclination of the plane is increased, show that the cone will slide before it topples over, if the coefficient of friction be less than $4\tan\alpha$.*

**Solution** When the cone is on the point of slipping, the inclination of the plane to the horizon is $\lambda$, given by

$$\mu = \tan\lambda \qquad \text{...(i)}$$




Let the plane be inclined at an angle $\phi$ to the horizon when the cone is on the point of toppling over. In this case the vertical line through the centre of gravity $G$ of the cone will just fall within the base of the cone *i.e.*, will pass through $L$.

$\therefore$ From $\Delta\, LGO$, we have $\tan\phi = \dfrac{LO}{OG}$ or $\tan\phi = \dfrac{r}{4h}$,

where $r$ is the radius of the base and $h$ is the height of the cone.

$\therefore \qquad \tan\phi = \dfrac{4r}{h} = 4\tan\alpha, \qquad \left[\because \tan\alpha = \dfrac{LO}{VO} = \dfrac{r}{h}\right]$

Now, the cone will slide before it topples over, if $\lambda < \phi$

*i.e.*, if $\tan\lambda < \tan\phi$ ...(ii)

*i.e.*, if $\mu < 4\tan\alpha$, [From Eqs. (i) and (ii)]

**Hence proved.**

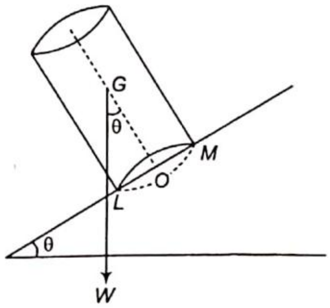
**Example 42.** *A uniform cylinder rests on its base on a rough inclined plane. As the inclination of the plane is increased gradually, then the cylinder will topple before it slides, if the ratio of the diameter of the base of the cylinder to its height is less than the coefficient of friction.*

**Solution** Let $\mu$ be the coefficient of friction between the base of the cylinder and the inclined plane.

When the cylinder is on the point of sliding, the inclination of the plane to the horizontal is equal to the angle of friction $\lambda$, where

$$\mu = \tan\lambda \qquad \text{...(i)}$$

Let $\theta$ be the inclination of the plane to the horizontal when the cylinder is on the point of toppling over. In this case the vertical line through the centre of gravity $G$ of the cylinder will just fall within the base of the cone, *i.e.*, will pass through $L$.




Let $r$ be the radius of the base and $h$ be the height of the cylinder.

Then $OG=\frac{1}{2}h$, where $O$ is the centre of the base of the cylinder.

$\therefore$ From $\Delta$ $LOG$, we get $\tan\theta=\frac{OL}{OG}=\frac{r}{1/2\,h}=\frac{2r}{h}$ ...(ii)

Now the cone will topple over before sliding, if $\lambda>\theta$

*i.e.*, if $\tan\lambda>\tan\theta$

*i.e.*, if $\mu>\frac{2r}{h}$ [From Eqs. (i) and (ii)]

*i.e.*, if $\frac{2r}{h}<\mu$

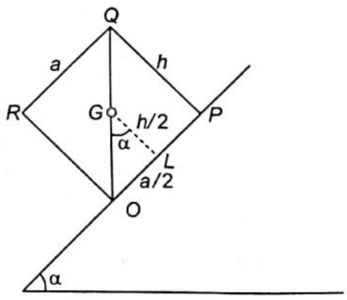
*i.e.*,

$\therefore$ $\frac{\text{Diameter of the base of the cylinder}}{\text{Height of the cylinder}}<\mu$ **Hence proved.**

**Example 43.** *A uniform rectangular block of height h whose base is a square of side a rests on a rough horizontal plane. The plane is gradually tilted about a line parallel to two edges of the base, then the block will slide or topple over according as $a>$ or $<\mu h$, where $\mu$ is the coefficient of friction.*

**Solution** Let rectangle $OPQR$ be the vertical section of the block through its centre of gravity $G$, where

$$OP=a \quad\text{and}\quad PQ=h$$




When the block is just on the point of toppling over, let $\alpha$ be the inclination of the plane to the horizontal and in this case the vertical line through $G$ must pass through $O$. From $G$ draw $GL$ perpendicular to $OP$.

Then in $\Delta$ $OGL$, we have

$$\tan\alpha=\frac{OL}{GL}=\frac{\frac{1}{2}a}{\frac{1}{2}h}=\frac{a}{h} \qquad \text{...(i)}$$

When the block is on the point of sliding, the inclination of the plane to the horizontal must be $\lambda$, given by

$$\mu=\tan\lambda \qquad \text{...(ii)}$$

$\therefore$ The block will slide or topple over according as $\lambda<$ or $>\alpha$

*i.e.*, $\tan\lambda<$ or $>\tan\alpha$

*i.e.*, $\mu<\text{or}>a/h$ [From Eqs. (i) and (ii)]

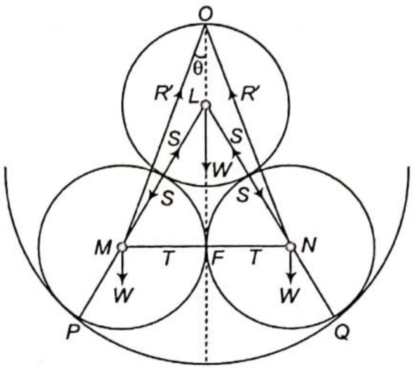
*i.e.*, $\mu h<\text{or}>a$

$\therefore$ $a>$ or $<\mu h$ **Hence proved.**

**Example 44.** *Inside a fixed hollow cylinder of radius R whose generators are horizontal are placed symmetrically two equal cylinders each of radius r a third cylinder equal to each of the latter is placed symmetrically on them, then the equilibrium cannot exist unless* $R < r(1 + 2\sqrt{7})$.

**Solution** Let $L, M, N$ are the centres of the smaller cylinders and $O$ that of the larger.

$\therefore$ $LMN$ is an equilateral triangle.




The upper cylinder is in equilibrium under the action of its weight $W$ (say) acting vertically downwards through $L$, and the reaction $S$ of each of the lower cylinders acting along $ML$ and $NL$ respectively.

Resolving these forces vertically, we have

$$2S\cos 30° = W$$

$$\Rightarrow \qquad S\sqrt{3} = W \qquad \ldots\text{(i)}$$

Consider the equilibrium of the cylinder with centre $M$. The forces acting on it are its weight $W$ acting vertically downwards through its centre $M$, the reaction $T$ of the cylinder with centre $N$ acting along the common normal in the sense $NM$, the reaction $S$ of the upper cylinder in the direction $LM$ and the reaction $R$ of the large cylinder in the sense of the common normal $PMO$.

Resolving horizontally these forces acting on the cylinder with centre $M$, we have

$$T + S\cos 60° = R\sin\theta \qquad \ldots\text{(ii)}$$

where, $\angle MON = 2\theta$

Consider the equilibrium of three smaller cylinders taken together. The forces acting on them when taken together are their combined weight $3W$ acting vertically downwards and the reactions $R$ of the larger cylinder on the cylinders with centre $M$ and $N$. Resolving these forces vertically, we have

$$2R\cos\theta = 3W \qquad \ldots\text{(iii)}$$

If the equilibrium exists, then $T$, the pressure between the lower cylinders must be the positive *i.e.*, $T \geq 0$.

*i.e.*, $R\sin\theta - S\cos 60° \geq 0$, [From Eq. (ii)]

or $\dfrac{3W}{2\cos\theta}\sin\theta - \dfrac{W}{\sqrt{3}}\cdot\dfrac{1}{2} \geq 0$ [From Eqs. (i) and (iii)]

or $$\tan\theta \geq \frac{1}{3\sqrt{3}} \qquad \ldots\text{(iv)}$$

Now, from figure, its is evident that $OM = R - r$ and $MF = r$

Therefore in $\Delta\, OMF$, we get $\sin\theta = \dfrac{MF}{OM}$

or $\sin\theta = \dfrac{r}{R-r}$ or $\tan\theta = \dfrac{r}{\sqrt{[(R-r^2)-r^2]}}$

$\therefore$ From Eq. (iv) we get

$$\frac{r}{\sqrt{[(R-r^2)-r^2]}} \geq \frac{1}{3\sqrt{3}} \quad \text{or} \quad \frac{r^2}{(R-r^2)-r^2} \geq \frac{1}{27} \quad \text{(By squaring both sides)}$$

or $27r^2 \geq (R-r)^2 - r^2$ or $27r^2 + r^2 \geq (R-r)^2$ or $28r^2 \geq (R-r)^2$

or $r\sqrt{28} \geq R - r$

or $R \leq r[2\sqrt{7} + 1]$

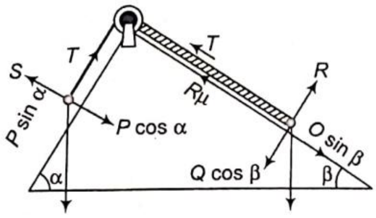
Hence, it is proved that equilibrium cannot exist unless

$$R \le r(1+2\sqrt{7}).$$

**Example 45.** *Two weights $P, Q$ resting on a double inclined plane are connected by a fine string passing over a common vertex and $Q$ is on the point of motion down the plane. Prove that the greatest weight which can be added to $P$ without disturbing equilibrium is $\dfrac{Q\sin 2\lambda \sin(\alpha+\beta)}{\sin^2\alpha - \sin^2\lambda}$;*

*$\alpha, \beta$ being the angles of inclination of the planes and $\lambda$ being the angle of friction.*

**Solution** As the weight $Q$ is on the point of moving down the plane so, the weight $P$ is on the point of moving up the plane. Hence, the forces of friction and normal reactions $R$ and $S$ are marked in the figure. Let $T$ be the tension in the string.




Resolving forces acting on the weight $Q$ along and perpendicular to the inclined plane on which $Q$ is placed, we have

$$T + \mu R = Q\sin\beta$$

and $$R = Q\cos\beta$$

Eliminating $R$ between these equations, we have

$$T = Q\sin\beta - \mu\, Q\cos\beta \quad \text{...(i)}$$

Similarly, for the other weight $P$, we have

$$T = P\sin\alpha + \mu\, P\cos\alpha$$

replacing $\mu$ by $-\mu$, $Q$ by $P$ and $\beta$ by $\alpha$ in Eq. (i)

Equating the values of $T$ from Eq. (i) and Eq. (ii),

we get $$Q\sin\beta - \mu Q\cos\beta = P\sin\alpha + \mu P\cos\alpha \quad \text{...(ii)}$$

or $$Q\left[\sin\beta - \frac{\sin\lambda}{\cos\lambda}\cos\beta\right] = P\left[\sin\alpha + \frac{\sin\lambda}{\cos\lambda}\cos\alpha\right]$$

$$\mu = \tan\lambda$$

or $$Q\sin(\beta-\lambda) = P\sin(\alpha+\lambda) \quad \text{...(iii)}$$

Let $V$ be the required greatest weight which can be added to $P$ such that the weight $(V+r)$ on the plane of inclination $\alpha$ is now on the point of moving downwards.

Then, from (iii) replacing $\beta$ by $\alpha$, $\alpha$ by $\beta$, $Q$ by $(V+P)$ and $P$ by $Q$, we get

$$(V+P)\sin(\alpha-\lambda) = Q\sin(\beta+\lambda) \quad \text{...(iv)}$$

or $$V\sin(\alpha-\lambda) = Q\sin(\beta+\lambda) - P\sin(\alpha-\lambda)$$

Multiplying both sides by $\sin(\alpha+\lambda)$ we get

$$V\sin(\alpha-\lambda)\sin(\alpha+\lambda) = Q\sin(\alpha+\lambda)\sin(\beta+\lambda) - P\sin(\alpha+\lambda)\sin(\alpha-\lambda).$$

or $$V(\sin^2\alpha - \sin^2\lambda) = \frac{1}{2}Q[2\sin(\alpha+\lambda)\sin(\beta+\lambda) - \sin(\beta-\lambda)\sin(\alpha-\lambda)]$$

$\{\ P\sin(\alpha+\lambda) = Q\sin(\beta-\lambda)\}$

$$= \frac{1}{2}Q\left[\{\cos(\alpha-\beta) - \cos(\alpha+\beta+2\lambda)\} - \{\cos(\beta-\alpha) - \cos(\beta+\alpha-2\lambda)\}\right]$$

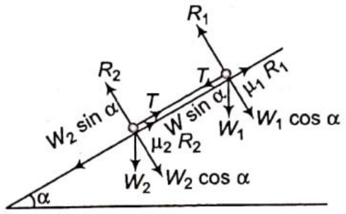
$$= \frac{1}{2}Q[\cos(\alpha+\beta-2\lambda) - \cos(\alpha+\beta+2\lambda)] = \frac{1}{2}Q[2\sin(\alpha+\beta)\sin 2\lambda]$$

or $$V = \frac{Q\sin(\alpha+\beta)\sin 2\lambda}{\sin^2\alpha - \sin^2\lambda}$$ **Hence proved.**

**Example 46.** *Two rough particles connected by a light string rest on an inclined plane. If their weights and corresponding coefficients of friction are $W_1, W_2$ and $\mu_1, \mu_2$ respectively, then the greatest inclination of the plane for equilibrium is*

$$\tan^{-1}(\mu_1 W_1 + \mu_2 W_2 / W_1 + W_2)$$

**Solution** Let the required greatest inclination of the plane be $\alpha$, then the particles would be on the point of sliding down the plane, hence the forces of friction $\mu_1 R_1$ and $\mu_2 R_2$ will be up the plane, where $R_1$ and $R_2$ are the normal reaction's of the plane on the particles of weight $W_1$ and $W_2$ respectively. Let $T$ be the tension in the string connecting the particles.




Resolving the forces acting on the particle of weight $W_1$ along and perpendicular to the inclined plane, we have

$$\mu_1 R_1 = W_1 \sin\alpha + T \quad \text{...(i)}$$

and $$R_1 = W_1 \cos\alpha \quad \text{...(ii)}$$

Similarly for the particle of weight $W_2$, we have $\mu_2 R_2 + T = W_2 \sin\alpha$ ...(iii)

and $$R_2 = W_2 \cos\alpha \quad \text{...(iv)}$$

From Eqs. (i) and (ii), we have

$$T = \mu_1 W_1 \cos\alpha - W_1 \sin\alpha \quad \text{...(v)}$$

From Eqs. (iii) and (iv), we have

$$T = W_2 \sin\alpha - \mu_2 W_2 \cos\alpha \quad \text{...(vi)}$$

Equating the values of $T$ from Eqs. (v) and (vi), we have

$$\mu_1 W_1 \cos\alpha - W_1 \sin\alpha = W_2 \sin\alpha - \mu_2 W_2 \cos\alpha$$

or $$(W_1 + W_2)\sin\alpha = (\mu_1 W_1 + \mu_2 W_2)\cos\alpha$$

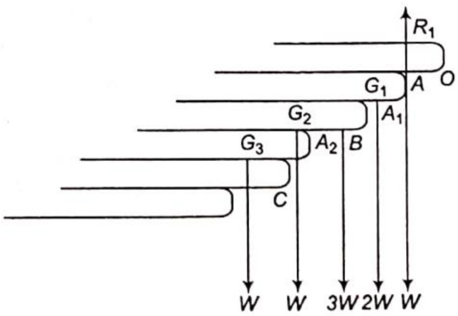
or $$\tan\alpha = \frac{\mu_1 W_1 + \mu_2 W_2}{W_1 + W_2}$$

or $$\alpha = \tan^{-1}\left(\frac{\mu_1 W_1 + \mu_2 W_2}{W_1 + W_2}\right)$$ **Hence proved.**

**Example 47.** *A pack of cards is laid on a table and each card projects in the directions of the length of the pack beyond the one below it; if each project as far as possible, show that the distance between the extremities of successive cards will form a harmonical progression.*

**Solution** Let $2a$ be the length of each card.

Consider the equilibrium of the uppermost card. The forces acting on this card are its weight $W$ acting at its centre of gravity $G_1$, such that $OG_1 = a$ and the reaction $R_1$ of the lower card. This reaction would act at $A$, since we are given that the cards are projected as far as possible, hence there would be geometrical contact only throughout the remaining portion of the uppermost card in contact with the lower one.



(a)

Now these two forces keep the uppermost card in equilibrium. Hence the reaction $R_1$ and the weight $W$ of the uppermost card will act along the same vertical line in opposite directions. Hence $G_1$ coincides with $A$

*i.e.,* $$OA = a \quad \text{...(i)}$$

Now consider the forces acting on the second card from upwards. Here the forces acting are, the weight $W$ of the card acting at its centre of gravity $G_2$, such that $AG_2 = a$ ; the weight $W$ of the first card acting at $A$ and the reaction $R_2$ of the third card (from upwards) on the second acting at $B$.

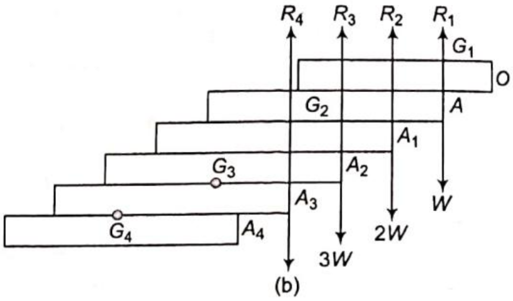
Now the first two forces together are equal to $2W$ acting at the mid-point $A$, of $AG_2$. Hence we are left with two forces *viz* the force $2W$ acting at $A_1$ and the reaction $R_2$ acting at $B$. These two forces keep the second card (from upwards) in equilibrium hence $R_2 = 2W$ and $B$ coincides with $A_1$ *i.e.*, $AA_1 = \frac{1}{2}a$.

Similarly, consider the equilibrium of the third card from upwards. The forces acting on this card are

(i) the weight $W$ of this card acting at $G_3$, such that $A_2G_3 = a$,

(ii) the weight $2W$ of the upper two cards acting at $A_1$,

(iii) the reaction $R_2$ of the fourth card (from upwards) on this card acting at $C$.



(b)

Now the first two forces are together equal

$2W + W$ *i.e.*, $3W$ acting at $A_2$, such that

$$\frac{A_1A_2}{A_2G_3} = \frac{1}{3}$$

or
$$A_1A_2 = \frac{1}{3}a$$

Thus we are left with two forces *viz* the forces $2W$ acting at $A_2$ and the reaction $R_3$ acting at $C$. Since these two forces keep the third card in equilibrium, so we have $R_3 = 3W$ and $C$ coincides with $A_2$.

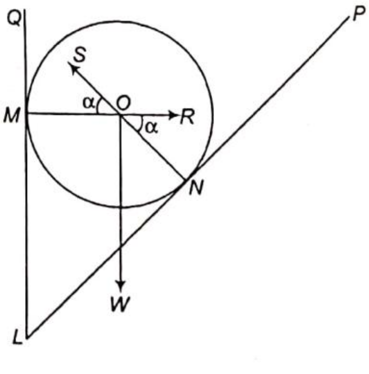
Similarly, considering the equilibrium of the fourth card, we can find $A_2A_3 = \frac{1}{4}a$.

Hence, the distances between the extremities of the successive cards are $\frac{1}{2}a, \frac{1}{3}a, \frac{1}{4}a, \ldots$ which evidently are in HP [Fig. (b)].

**Example 48.** *A sphere of weight W is placed between two smooth planes, one of which is vertical and the other inclined at an angle α to the vertical. Find the reactions of the planes.*

**Solution** The sphere is in equilibrium under the action of three forces *viz*, its weight $W$ acting, vertically downwards through its centre $O$ and the reactions $R$ and $S$ acting at the points of contact $M$ and $N$ of the sphere with the given planes.

From the figure its is evident that




$$\angle RON = \alpha = \angle MOS$$

$\therefore$
$$\angle SOR = 180° - \alpha$$

and
$$\angle SOW = 90° + \alpha,$$

Applying Lami's theorem at $O$, we get

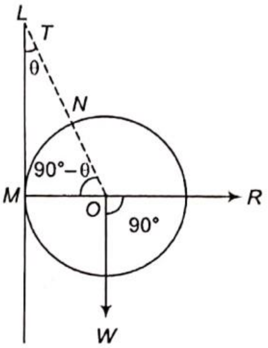
$$\frac{R}{\sin SOW} = \frac{S}{\sin ROW} = \frac{W}{\sin SOR}$$

$$\Rightarrow \quad \frac{R}{\sin(90° + \alpha)} = \frac{S}{\sin 90°} = \frac{W}{\sin(180° - \alpha)}$$

$$\Rightarrow \quad \frac{R}{\cos\alpha} = \frac{S}{1} = \frac{W}{\sin\alpha}$$

$\Rightarrow$ $R = W\cot\alpha$ and $S = W\operatorname{cosec}\alpha$

**Example 49.** *A smooth sphere is supported in contact with a smooth vertical wall by a string fastened to a point on its surface, the other end being attached to a point in the wall; if the length of the string be equal to the radius of the sphere, find the inclination of the string to the vertical, the tension of the string and the reaction of the wall.*




**Solution** $LM$ is the wall and $O$ is the centre of the sphere.

The forces acting on the sphere are

(i) its weight $W$ acting vertically downwards at $O$,

(ii) the normal reaction $R$ between the sphere and the wall at the point of contact $M$ acting normally to the wall, passing through the centre $O$,

(iii) the tension $T$ in the string $LN$.

Since, two forces $R$ and $W$ are meeting at $O$, therefore the third force $T$ is also passing through $O$.

Let $\theta$ be the inclination of the string to the vertical. Also we are given that (string $LN$ = radius $ON$)

*i.e.*, $$OL = ON + NL = 2\,ON$$

$\therefore$ In $\Delta\, OML$,

$$\sin\theta = \frac{OM}{OL} = \frac{OM}{2\,ON} = \frac{1}{2} \qquad \ldots(i)$$

($\because OM = ON =$ radius)

Applying Lami's theorem at $O$, we get

$$\frac{T}{\sin 90^\circ} = \frac{R}{\sin[(90^\circ + (90^\circ - \theta)]} = \frac{W}{\sin[180^\circ - (90^\circ - \theta)]}$$

$$\Rightarrow \qquad \frac{T}{1} = \frac{R}{\sin\theta} = \frac{W}{\cos\theta} \qquad \ldots(ii)$$

Now, from Eq. (i),

$$\sin\theta = \frac{1}{2},$$

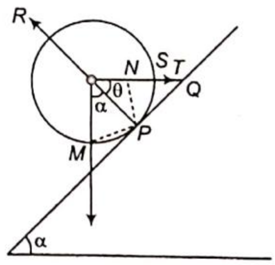
$$\therefore \qquad \cos\theta = \frac{\sqrt{3}}{2} \text{ and } \tan\theta = \frac{1}{\sqrt{3}}$$

Therefore, from Eq. (ii)

$$T = \frac{W}{\cos\theta} = \frac{2W}{\sqrt{3}} \text{ and } R = W\tan\theta = \frac{W}{\sqrt{3}}$$

**Example 50.** *A sphere of weight $W$ and radius $a$ is kept in equilibrium on a smooth inclined plane by means of a string of length $l$. One end of the string is attached to the sphere and the other to the inclined plane. If the plane is inclined to the horizontal at an angle $\alpha$, then show that the tension in the string is*

$$[W(a + l)\sin\alpha]\sqrt{(2al + l^2)}$$




**Solution** In the position of equilibrium $P$ is the point of contact of the inclined plane and the sphere with centre $O$.

The sphere in this position is in equilibrium under the action of three forces, *viz*,

(i) its weight $W$ acting vertically downwards through its centre $O$,

(ii) the reaction $R$ between the sphere and the plane acting at right angles to the inclined plane at $P$ and passing through $O$,

(iii) the tension $T$ in the string $SQ$, whose one end is attached at $S$ on the sphere and the other at $Q$ on the inclined plane.

Since, lines of action of two of these process *viz*, $R$ and $W$ meet at $O$, so the line of action of the third force $T$ will also pass through $O$.

Let $\angle POQ = \theta$ and from the figure above it is evident that $\angle POM = \alpha$.

From $P$ draw $PM$ and $PN$ perpendiculars to lines of action of the forces $W$ and $T$ respectively.

Taking moments about $P$, we get

$$T \cdot PN = W \cdot PM$$

or
$$T \cdot OP \sin\theta = W \cdot OP \sin\alpha$$

or
$$T = W(\sin\alpha / \sin\theta) \quad \text{...(i)}$$

Also in $\Delta\, POQ$, we find that

$$\cos\theta = \frac{OP}{OQ} = \frac{OP}{OS + SQ} = \frac{a}{a+l} \quad \text{...(ii)}$$

where $a$ is the radius of the sphere and $l$ the length of the string.

Also
$$\sin\theta = \sqrt{(1-\cos^2\theta)} = \sqrt{[1-\{a/(a+l)\}^2]} \quad \text{...(ii)}$$

$$= \sqrt{\{(a+l)^2 - a^2\}/(a+l)^2} = \sqrt{a^2 + l^2 + 2al - a^2} \cdot \frac{1}{(a+l)}$$

$$\sin\theta = \sqrt{(2al + l^2)}/(a+l)$$

Putting this value of $\sin\theta$ in Eq. (i), we get

$$T = \frac{(W\sin\alpha)(a+l)}{\sqrt{(2al+l^2)}}$$

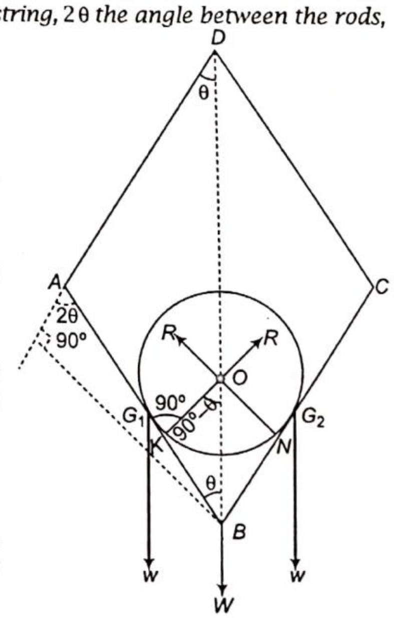
or the tension in the string $SQ$ is

$$= \frac{[W(a+l)\sin\alpha]}{\sqrt{(2al+l^2)}}$$

**Hence proved.**

**Example 51.** *Two equal rods of weight $w$ are freely joined. Their free ends are attached by string to a fixed point. A circular disc of weight $W$ and radius $r$ rest in the angle between the rods and the whole hangs in a vertical plane. If $2a$ be the length of each rod and each string, $2\theta$ the angle between the rods, prove that*

$$r = 2a\sin^2\theta\tan\theta\left(\frac{3w+2W}{W}\right)$$




**Solution** $AB$ and $BC$ are the rods; $AD$ and $CD$ are the strings. The weight $w$ of the rods are acting $G_1$ and $G_2$, the mid-points of $AB$ and $BC$ respectively. The weight $W$ of the disc is acting at its centre $O$.

$T$ is the tension in the strings $AD$ and $CD$.

Resolving vertically the forces acting on the system as a whole, we get

$$2T\cos\theta = W + 2w \quad \text{...(i)}$$

The disc is in equilibrium under the action of its weight $W$ and the reactions $R$ of the rods at the points of contact $K$ and $N$ of the disc and the rods. Resolving these forces vertically we get

$$2R\cos(90° - \theta) = W$$

or
$$2R\sin\theta = W \quad \text{...(ii)}$$

The rod $AB$ is in equilibrium under the action of its weight $w$ at $G_1$, tension $T$ at $A$, reaction $R$ of the disc at $K$ and the reaction between the rods at $B$.

Taking moment of these forces about $B$, we have

$$G \cdot BL = w\, BG_1 \sin\theta + R \cdot BK$$

or $$T \cdot AB \sin 2\theta = w \frac{1}{2} AB \sin\theta - R \cdot OB \cos\theta \qquad \text{(See figure)}$$

or $$T \cdot 2a \sin 2\theta = w \cdot a \sin\theta + R \cdot r \cot\theta \qquad (AB = 2a \text{ and } OB = r \operatorname{cosec}\theta)$$

or $$\left(\frac{W + 2w}{2\cos\theta}\right) \cdot 2a \sin 2\theta = w \cdot a \sin\theta + \left(\frac{W}{2\sin\theta}\right) \cdot r \cot\theta \qquad \text{[From Eqs.(i) and (ii)]}$$

or $$2a\,(W + 2w)\sin\theta = w \cdot a \sin\theta + \frac{Wr\cos\theta}{2\sin^2\theta}$$

or $$Wr\cos\theta = 2\sin^2\theta\ [\{2a\,(W + 2w)\sin\theta - (w \cdot a)\sin\theta]$$

or $$Wr\cos\theta = 2\sin^2\theta\ (2aW\sin\theta + 4aw\sin\theta - wa\sin\theta)$$

or $$Wr\cos\theta = 2\sin^2\theta\ (2aW\sin\theta + 3aw\sin\theta)$$

or $$Wr\cos\theta = 2\sin^2\theta \cdot \sin\theta\ (2aW + 3aw)$$

or $$r = \frac{2\sin^2\theta \cdot \sin\theta\,[2aW + 3aw]}{\cos\theta \cdot W}$$

or $$r = 2\sin^2\theta \cdot \tan\theta \cdot a\ \left(\frac{2W + 3w}{W}\right)$$

or $$r = 2a\sin^2\theta\tan\theta\left(\frac{2W + 3w}{W}\right)$$ **Hence proved.**

**Example 52.** *Two blocks of masses m and M are connected by a chord passing around a frictionless pulley which is attached to a rotating frame, which rotates about a vertical axis with an angular velocity $\omega$.*
*If the coefficient of friction between the two masses and the surface be $\mu_1$ and $\mu_2$ respectively, determine the value of $\omega$, at which the block starts sliding radially ($M > m$).*

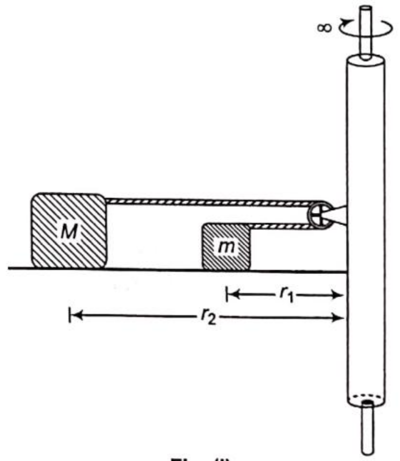



Fig. (i)

**Solution** Evidently, the larger block of mass $M$ experiences more centrifugal force radially outwards, compared to the block of smaller mass $m$, ($M > m$ and $r_2 > r_1$)

Fig. (ii) shows their FBD

Owing to the larger force experienced by block of mass $M$, it tends to fly off radially.

In the situation of limiting equilibrium, we have,

$T = m\omega^2 r_1 + f_1$ and $T + f_2 = M\omega^2 r_2$

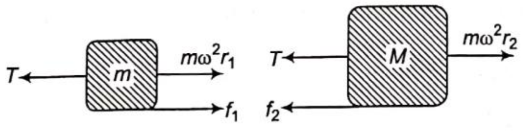



Fig. (ii)

(where $f_1$ and $f_2$ are frictional forces for the two blocks and the surface)

$$f_1 = \mu_1 mg \text{ and } f_2 = \mu_2 Mg$$

$\therefore$ The above two equations get reduced to

$$T = m\,\omega^2 r_1 + \mu_1 mg \qquad \ldots\text{(i)}$$

and $$T + \mu_2 Mg = M\omega^2 r_2 \qquad \ldots\text{(ii)}$$

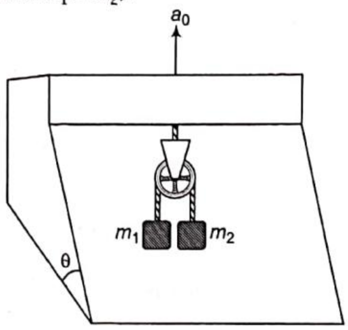
Subtracting Eq. (i) from Eq. (ii)

$$\mu_2 Mg = M\omega^2 r_2 - m\omega^2 r_1 - \mu_1 mg$$

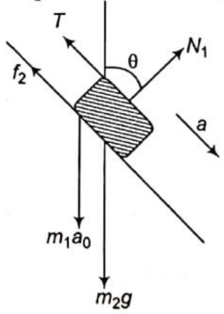
$$\Rightarrow \quad \omega^2 = \frac{g(\mu_1 m + \mu_2 M)}{Mr_2 - mr_1}$$

$$\Rightarrow \quad \omega = \left[\frac{g(\mu_1 m + \mu_2 M)}{Mr_2 - mr_1}\right]^{1/2}$$

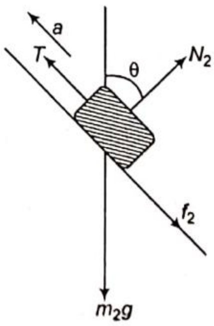
**Example 53.** *Two blocks lying on a rough inclined plane are connected by a light string. If the plane be given an acceleration $a_0$ vertically upwards, find the acceleration of the block $m_1$ relative to the incline. (Assume $\mu$ = coefficient of friction and $m_1 > m_2$).*




**Solution** Let us attach a reference frame to the inclined plane which would be a non-inertial frame of reference. The acceleration of the block $m_1$ found thus, would be the required acceleration.




Both the blocks will be acted upon by pseudo forces and further, since block of mass $m_1$ is heavier compared to the other, it would tend to descend down the plane and hence friction would be upwards along the plane.



Hence, we have

$$N_1 = m_1 (a_0 + g)\cos\theta$$

$$m_1 (a_0 + g)\sin\theta - T - f_1 = m_1 a$$

$$\Rightarrow \quad m_1 (a_0 + g)\sin\theta - T - \mu m_1 (a_0 + g)\cos\theta = m_1 a \qquad \text{...(i)}$$

Similarly,

$$N_2 = m_2 (a_0 + g)\cos\theta$$

$$T - f_2 - m_2 (a_0 + g)\sin\theta = m_2 a$$

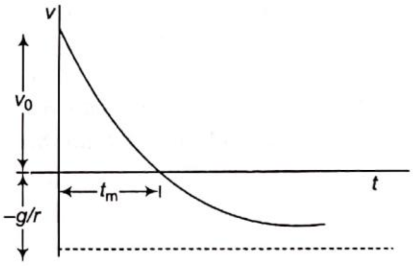
$$\Rightarrow \quad T - m_2 \mu (a_0 + g)\cos\theta - m_2 (a_0 + g)\sin\theta = m_2 a \qquad \text{...(ii)}$$

Eq. (i) +Eq. (ii) $\Rightarrow$

$$a = \frac{(a_0 + g)(m_1 - m_2)\sin\theta - \mu (m_1 + m_2)\cos\theta}{m_1 + m_2}$$

**Example 54.** *An object is thrown upward with an initial velocity $v_0$. The drag on the object is assumed to be proportional to the velocity. What time will it take the object to more upward and what maximal altitude will it reach?*

**Solution** Since, the drag is proportional to the velocity of the object, so is the acceleration caused by this force (with a minus sign). Hence, by Newton's second law,




$$\frac{dv}{dt} = -g - rv$$

where $r$ is the proportionally factor, whence

$$\int_{v_0}^{v} \frac{dv}{v + g/r} = -r\int_0^t dt$$

Integration yields

$$v = \left(v_0 + \frac{g}{r}\right)e^{-rt} - \frac{g}{r} \qquad \text{...(i)}$$

For $v = 0$ this yields

$$t_m = \frac{1}{r}\ln\left(1 + \frac{rv_0}{g}\right) \qquad \text{...(ii)}$$

To find the maximal altitude, we rewrite Eq.(i) in the form

$$\frac{dh}{dt} = \left(v_0 + \frac{g}{r}\right)e^{-rt} - \frac{g}{r} \qquad \text{...(iii)}$$

Integrating this equation to $t$, we find that

$$h = \left(v_0 + \frac{g}{r}\right)\frac{1}{r}(1 - e^{-rt}) - \frac{g}{r}t \qquad \text{...(iv)}$$

Bearing in mind that at that point of greatest ascent $v = \frac{dh}{dt} = 0$ and combing this result with Eq. (iii), we get

$$\left(v_0 + \frac{g}{r}\right)e^{-rt_m} = \frac{g}{r} \qquad \text{...(v)}$$

Substituting $t_m$ from Eq.(ii), we arrive at the final result

$$h = \frac{1}{r}\left[v_0 - \frac{g}{r}\ln\left(1 - \frac{rv_0}{g}\right)\right].$$

When drag is extremely low, or $r v_0/g << 1$, we can employ the expansion

$$\ln\left(1-\frac{rv_0}{g}\right) \approx \frac{rv_0}{g} - \frac{1}{2}\left(\frac{rv_0}{g}\right)^2$$

This results in the well-known formula,

$$h = \frac{v_0^2}{2g}.$$

**Example 55.** *A tractor pulls a sledge loaded with logs over an icy road at a constant speed of 15 km/h. With what speed will the tractor pull the same sledge and its load in summer over a ledger road if the power developed by the engine is the same in both cases? The coefficient of friction for motion over the icy road is $k_1 = 0.01$ and over the ledger road $k_2 = 0.15$.*

**Solution** Since, the power of the engine is the same in both cases, the following relation should hold.

$$N = F_1 v_1 = F_2 v_2 \qquad \text{...(i)}$$

where $F_1$ and $v_1$ are the tractive force of the engine and the speed of the tractor on the icy road; $F_2$ and $v_2$ are the tractive force and the speed over the ledger road.

Since at constant speed the work done by the tractive force of the engine is only expanded in both cases to overcome the forces of friction,

$$F_1 = k_1 P \text{ and } F_2 = k_2 P \qquad \text{...(ii)}$$

where $P$ is the weight of the sledge.

It follows from Eqs. (i) and (ii) that

$$k_1 v_1 = k_2 v_2 \Rightarrow v_2 = \frac{k_1 v_1}{k_2} \qquad \text{...(iii)}$$

Now, according to question.
we have,

$$k_1 = 0.01,\ v_1 = 15 \text{ km/h and } k_2 = 0.15$$

$\therefore$ Putting these values in Eq.(iii), we get

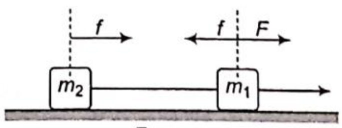
$$v_2 = \frac{0.01 \times 15}{0.15}$$

$$\Rightarrow \qquad v_2 = 1 \text{ km/h.}$$

$\therefore$ The tractor will pull the same sledge and its load in summer over a ledger road with speed of 1km/h.

**Example 56.** *A man standing on the platform of a decimal balance performs rapid squatting motions. How will the readings of the balance change at the beginning and end of squatting?*

**Solution** When the man begins to squat he relaxes the muscles in his legs and allows his body to "fall" with a certain acceleration $a$ directed downwards and the pressure on the platform of the balance $f$ becomes such that,




$$mg - F = ma$$

or $\qquad F = mg - ma$

i.e., $\qquad F < mg$

At the end of the squatting the man increases the tension in the muscles of his legs, thus increasing the pressure on the platform and creating the acceleration $a$ directed upwards which is necessary to compensate for the velocity acquired during squatting. In this case the second law equation will take the form

$$F - mg = ma$$

and the pressure will be,

$$F = mg + ma$$

*i.e.,* $$F > mg \qquad \ldots\text{(ii)}$$

Hence, at the beginning of the squatting motions $F < mg$ and at the end $F > mg$.

**Example 57.** *Weights $P_1$ and $P_2$ are connected by a thread string over a fixed pulley block. Initially the centres of gravity of the weights are at the same height. Determine with what acceleration and in what vertical direction the centre of gravity of the combination of weights will move if $P_1 > P_2$.*

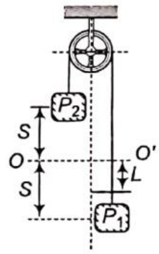
**Solution** After a time $t$ each weight will be displaced from the initial position by a distance

$$S = \frac{at^2}{2}$$

(By the equation of Newton's second law)

where, $$a = \frac{P_1 - P_2}{P_1 + P_2} g$$

In this case the centre of gravity of the system should obviously move down a certain distance $L$ from the initial position towards the larger weight (see figure). Upon determining the centre of gravity, the distances of the centre of gravity from the weights should be inversely proportional to the magnitudes of these weights




*i.e.,* $$\frac{S+L}{S-L} = \frac{P_1}{P_2}$$

or $$L = S\,\frac{\frac{P_1}{P_2} - 1}{1 + \frac{P_1}{P_2}} = \frac{P_1 - P_2}{P_1 + P_2} S \qquad \ldots\text{(i)}$$

Inserting the value of $S = \frac{at^2}{2}$ in Eq.(i), we get

$$L = \frac{P_1 - P_2}{P_1 + P_2} \times \frac{at^2}{2} \qquad \ldots\text{(ii)}$$

Comparing this result with the formula for the path of uniformly accelerated motion and then inserting the values of the acceleration of the weights we find that the centre of gravity should move down with an acceleration

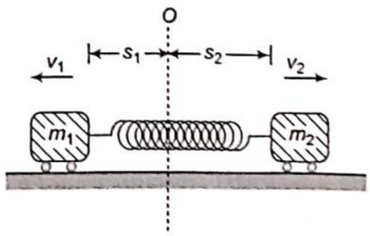
$$j = \frac{P_1 - P_2}{P_1 + P_2} a = \left(\frac{P_1 - P_2}{P_1 + P_2}\right)^2 g$$

The magnitude of the acceleration of the centre of gravity is less than that of each weight separately.

**Example 58.** *A compressed spring is situated between two carts of mass $m_1$ and $m_2$. When the spring expands and assumes its initial state it acts on each cart with an average force F for a time $\tau$. Then, after the spring is fully expanded and its action ceases. Prove that the carts will more along horizontal rails in such a way that their centre of mass (centre of gravity) remains at rest. Disregard friction.*

**Solution** The combined centre of mass of the carts before motion will lie on the straight line $OO'$. By definition the distances of the carts $l_1$ and $l_2$ to their centre of mass should at any moment of time $t$ be inversely proportional to the masses of the carts.

*i.e.,* $$\frac{l_1}{l_2} = \frac{m_2}{m_1}$$




The distances covered by the carts during the time $t$ will be

$$s_1 = v_1 t \text{ and } s_2 = v_2 t$$

*i.e.,*
$$\frac{s_1}{s_2} = \frac{v_1}{v_2}$$

The velocities imparted to the carts by the action of the compressed spring will, by Newton's second law, be equal to

$$v_1 = \frac{F\tau}{m_1} \text{ and } v_2 = \frac{F\tau}{m_2}$$

Therefore,

$$\frac{s_1}{s_2} = \frac{v_1}{v_2} = \frac{F\tau}{m_1} \cdot \frac{m_2}{F\tau} = \frac{m_2}{m_1} = \frac{l_1}{l_2}$$

and the distances of the carts from the straight line $OO'$ satisfy the same ratio as the distances from the centre of the gravity. *i.e.,* the centre of gravity of the carts is always on the straight line $OO'$.

This result can also be obtained directly from the law of conservation of momentum which gives in this case

$$m_1 v_1 = m_2 v_2 \text{ or } \frac{v_1}{v_2} = \frac{m_2}{m_1}$$

**Example 59.** *Assume that the jet engine of a rocket ejects the products of combustion in portions whose masses are $m = 200$ g and whose velocity on exit from the nozzle of the engine is $v = 1000$ m/s.*

*Assuming it flies horizontally with what velocity will rocket move after the ejection of the third portion of the gas? What will the velocity of the rocket be at the end of the first second of motion if the engine produces twenty bursts per second.*

*The initial mass of the rocket is $M = 300$ kg and its initial velocity is zero. Disregard the air resistance to the motion of the rocket.*

**Solution** Denoting the velocity of the rocket by $v_1$ after the ejection of the first portion of gas, by $v_2$ after the second portion, by $v_3$ after the third and by $v_N$ after the $N$th portion and utilizing the law of conservation of momentum, we obtain

For the velocity of the rocket after the ejection of the first portion of gas

$$(M - m)v_1 - mv = 0 \text{ or } v_1 = \frac{mv}{M - m} \quad \text{...(i)}$$

After the second portion

$$(M - m)v_1 = (M - 2m)v_2 - mv \text{ or } v_2 = \frac{2mv}{M - 2m} \quad \text{...(ii)}$$

After the third portion

$$(M - m)v_2 = (M - 3m)v_3 - mv$$

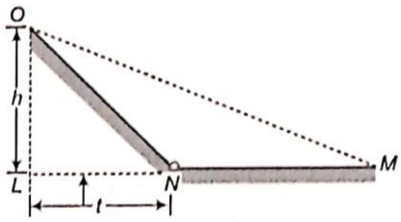
or
$$v_3 = \frac{3mv}{M - 3m} = 2 \text{ m/s} \quad \text{...(iii)}$$

Therefore, the velocity of the rocket after $N^{\text{th}}$ ejection is

$$v_N = \frac{Nmv}{M - Nm}; \; v_{20} = 13.5 \text{ m/s}$$

**Example 60.** *A sledge slides down an icy hill of height h (by figure) and stops after covering a distance NM. The distance LM is equal to s.*

*Determine the coefficient of friction k between the sledge and the icy surface. Calculate the acceleration of the sledge over the path ON and over the path NM.*




**Solution** The sledge at the top of the hill has a potential energy

$$E = mgh$$

During motion this energy is expanded on the work $L_1$ to overcome the forces of friction over the path $ON$ and on the work $L_2$ to overcome these forces over the path $NM$,

*i.e.*, $$E = mgh = L_1 + L_2$$

The force of friction $F_1$ over the path $ON$

$$F_1 = kmg\frac{l}{\sqrt{l^2+h^2}}$$

where $l$ is the length of $LN$. The work will be

$$L_1 = F_1\, ON = klmg$$

For the path $NM$ the force of friction

$$F_2 = kmg \text{ and the work is}$$

$$L_2 = F_2\, MN = kmg\,(s-l)$$

Hence, $$mgh = L_1 + L_2 = mgks \text{ and } k = \frac{h}{s}$$

The equation of Newton's second law for the motion of the sledge over the path $ON$ will be

$$mg\frac{h}{\sqrt{l^2+h^2}} - F_1 = ma_1 \text{ and therefore } a' = \frac{gh}{\sqrt{l^2+h^2}}\left(l - \frac{l}{s}\right)$$

Since $\frac{l}{s} < 1, a > 0$ and the sledge will move over the path $ON$ with a uniformly accelerated motion.

The acceleration over the path $MN$ is $a' = -kg$ and the sledge moves with a uniformly retarted motion.

**Example 61.** *A man in a boat A, of mass $m_1 = 300$ kg pulls a rope with a force $F = 10$ kgf. The other end of the rope is tied first to a tree on the bank and then to a boat B of mass $m = 200$ kg.*
*Determine the velocity of the boat A in both cases at the end of the third second. What work will be done in this time and what power will the man develop in these two cases by the end of the third second?*
*Disregard the weight of the rope and the resistance of the water.*

**Solution** In both cases the man imparts the same acceleration.

$$a = \frac{F}{m_1}$$

To the boat $A$ and therefore the velocity of the boat $A$ $v_1 = at = \frac{F}{m_1}t = 1$ m/s

will be the same in both cases.
In the first case the work done by the man is

$$A_1 = \frac{m_1 v_1^2}{2} = 15 \text{ kgf-m}$$

and in the second case

$$A_2 = \frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2} = 37.5 \text{ kgf-m}$$

Where $v_2 = \frac{F}{m_2}t$ is the velocity of the second boat by the end of the third second. The power developed by the man at the end of the third second in the first case is

$$N_1 = Fv_1 = 10 \text{ kgf-m/s}$$

and in the second

$$N_2 = F(v_1 + v_2) = 25 \text{ kgf-m/s.}$$

If $\qquad N_2 > N_1$

**Example 62.** *An hour glass consists of two compartments joined end to end by a connecting tube through which sand from upper compartment can fall to the lower compartment.*

*If the hour glass is freshly inverted and placed on a weighing pan, describe with the aid of a graph how the reading on the weighing scale would change until all the sand has dropped to the bottom compartment.*

**(Ist NPO Singapore)**

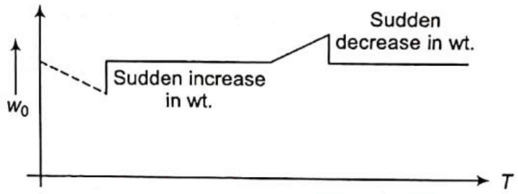
**Solution** In the steady state when all sand has settled down in the lower compartment the scale reading is obviously equal to the weight of the hour glass plus sand ($w_0$).

When the hour glass is freshly inverted sand begins to transit from the top to the bottom compartment.

The position of sand in transit is free falling and hence contribute no weight until it hits the bottom compartment. Thus initially the pan will show a steady decrease in weight until the first sand particles hits the bottom compartment.

When sand hits the bottom compartment its velocity is instantly reduced to zero. This is translated as an impulsive force equal to the rate of change of momentum experienced by the bottom compartment and hence a sudden (step wise) increase in weight is registered.

During the interval when the string of sand in transit is constantly falling and hitting the bottom compartment and being replenished by more sand falling from the top compartment, the system is in a steady state and the net weight of the system is equal to $w_0$, *i.e.*, as the falling sand string had been frozen in motion. The rate of loss in weight by the top compartment is just balanced by the rate of gain in weight by the bottom one.




When the top compartment just becomes empty, there is no further loss in weight of the top compartment while the bottom compartment continues to gain more sand *i.e.*, more weight at a steady state.

The gain ends when the last bit of sand has fallen down, at this instant the weighing pan suddenly losses its supply of falling sand and its associated impulsive force. There is thus a step wise drop in reading to the find steady value of $w_0$.

**Example 63.** *An amusement park ride consists of a large vertical cylinder that spins about its axis fast enough that any person inside is held up against the wall when the platform is removed. The coefficient of static friction between the wall and person is 0.3 and radius of cylinder is 5.0 m. Find the maximum period of revolution necessary to prevent the person from falling. How many revolutions per second does the cylinder perform?*

**(NSEP 2001-2002)**

**Solution** Maximum upward force due to static friction $F_{max} = \mu \times$ pressing force

$F_{max} = \mu \times mr\omega^2$ to prevent the person from falling, we require

$$mg \le \mu\, mr\, \omega^2$$

$$\omega^2 \ge \frac{g}{\mu r}, \quad \frac{4\pi^2}{T^2} \ge \frac{g}{\mu r}$$

or

$$T \le 2\pi\sqrt{\frac{\mu r}{g}}$$

$$T_{max} = 2\pi\sqrt{\frac{\mu r}{g}} = 2\pi\sqrt{\frac{0.5 \times 5}{9.8}} = 2.46 \text{ s}$$

Revolution per minute $= \dfrac{60}{2.46} = 24.4$ pm

**Example 64.** *A force F acts on an object which is initially at rest. The graph shows how the magnitude of the force veries with time t.*

(*a*) *Sketch a graph showing how the velocity of the object would vary during the time for which the force acts.*

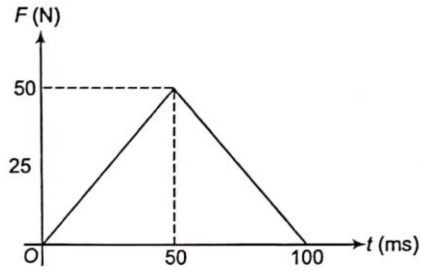
(*b*) *Calculate the momentum of the object after* 50 *ms.*

(*c*) *Find the average value of the force.*

**(British PO 1995)**

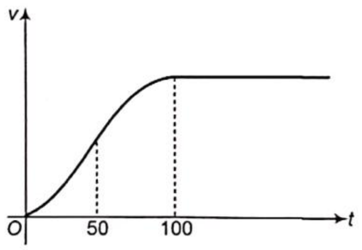
**Solution**

(a)



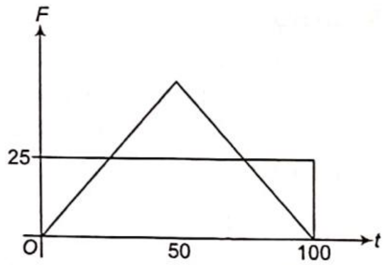

(b) Impulse = Force × time

= change of momentum = area of triangle




$= \frac{1}{2} \times 50 \times 100$

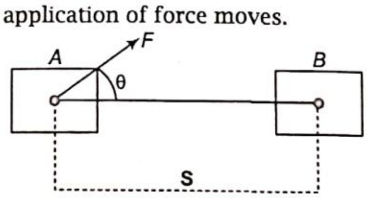
(c) From rectangular area = triangular area average force = 25 N




## Work, Energy and Power

### Work

Work is done when the point of application of force moves.




$W = \mathbf{F} \cdot \mathbf{S} = FS \cos \theta$

### Special cases

1. If $\theta = 0°, \cos \theta = 1$

   $\Rightarrow$ **F** and **S** in same direction.

   $$W = FS$$

2. If $0° < \theta < 90°$, $\cos\theta > 0$

   $\Rightarrow$ **F** has a component in the direction of **S**.

   So, $W$ is +ve.

3. If $\theta = 90°$, $\cos\theta = 0$

   $\Rightarrow$ **F** is perpendicular to **S**.

   $$W = 0$$

4. If $90° < \theta < 180°$, $\cos\theta < 0$

   **F** has a component in the direction opposite to **S**.

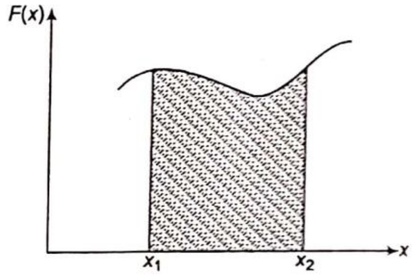
   So, $W$ is - ve.

5. If $\theta = 180°$, $\cos\theta = -1$

   **F** and **S** are in opposite direction.

   $W = -FS$

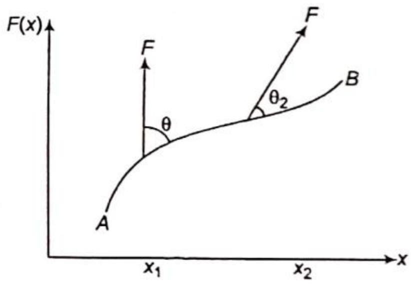
   Work done by a variable force




$$W = \int_{x_1}^{x_2} F(x)dx$$

$\Rightarrow$ Work is the area under **F** and **S** curve.

$\Rightarrow$ When force changes magnitude and direction from point to point



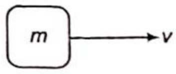

$$W_{AB} = \int_A^B F\,dS$$

## Energy

Capacity to do work

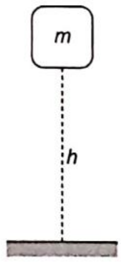
Mechanical energy = Kinetic energy + Potential energy

(a) Kinetic energy $= \frac{1}{2}Mv^2$




(b) Potential energy

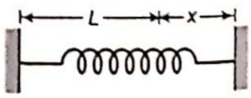
1. PE of a body of mass $m$ placed at a vertical height $h$




$$\text{PE} = mgh$$

2. PE of a spring when it is elongated or compressed by an external force

$$\text{PE} = \frac{1}{2}kx^2$$




$\Rightarrow$ Mechanical energy of a body in a conservative force field remain constant

$$(\text{KE})_i + (\text{PE})_i = (\text{KE})_f + (\text{PE})_f$$

## Power

The rate at which work is done by a body or a system.

Average power $P = \dfrac{\Delta W}{\Delta t}$

Instantaneous power $P_t = \lim\limits_{\Delta t \to 0} \left(\dfrac{\Delta W}{\Delta t}\right)$

$$P_t = \frac{dW}{dt}$$

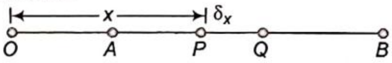
⇒ Power is also defined as scaler product of force and velocity.

$$P = \mathbf{F} \cdot \mathbf{v}$$

# Illustrative Solved Examples

**Example 65.** *Prove that the work done against the tension in stretching a light extensible string is equal to the product of its extension and the mean of the initial and final tensions.*

**Solution** Let $L$ be the natural length of the string.




Let it be extended from $A$ to $B$ such that $OA = a$ and $OB = b$. We are to find the work done against the tension in stretching the string from $A$ to $B$. Let $\lambda$ be the modulus of elasticity of the string.

Let $\quad OP = x$ and $PQ = \delta x$

$PQ$ being small, tensions at $P$ and $Q$ can be taken to remain same.

Tension in the string when it is stretched upto $P = \dfrac{\lambda}{L}(x - L)$ acting in the direction $PO$.

$\therefore$ Work done $= \left[\dfrac{\lambda}{L}(x - L)\right]\delta x$ ( In case of $PQ$)

$\therefore$ Work done against tension in stretching from $A$ to $B = \displaystyle\int_{x=a}^{b} \frac{\lambda}{L}(x - L)\,dx$

$$= \frac{\lambda}{L}\left[\frac{1}{2}(x - L)^2\right]_a^b = \frac{1}{2}\frac{\lambda}{L}[(b - L)^2 - (a - L)^2] \qquad \ldots(\text{i})$$

$$= \frac{1}{2}\frac{\lambda}{L}(b - a)[(b - L) + (a - L)]$$

$$= (b - a)\frac{1}{2}\left[\frac{\lambda}{L}(b - L) + \frac{\lambda}{L}(a - L)\right] \qquad \ldots(\text{ii})$$

Also if $T_1$ and $T_2$ be the tensions in the string at $A$ and $B$ then we have

$$T_1 = \frac{\lambda}{L}(a - L) \text{ and } T_2 = \frac{\lambda}{L}(b - L) \qquad \ldots(\text{iii})$$

Also extension produced $= AB = b - a$

$\therefore$ From Eq. (ii) we find that work done against the tension in stretching from $A$ to $B$

$$= (\text{Extension produced}) \times \frac{1}{2}(T_1 + T_2)$$

$$= (\text{Extension produced}) \times (\text{mean of initial and final tensions})$$

**Example 66.** *A uniform elastic string has length $a_1$ when the tension is $T_1$ and a length $a_2$ when the tension is $T_2$. Show that its natural length is $(a_2T_1 - a_1T_2)/(T_1 - T_2)$ and that the amount of work done in stretching it from natural length to $(a_1 + a_2)$ is $\dfrac{1}{2}\dfrac{(a_1T_1 - a_2T_2)^2}{(T_1 - T_2)(a_1 - a_2)}$*

**Solution** Let $L$ be the natural length of the string and $\lambda$ its modulus of elasticity.

$$T_1 = \frac{\lambda}{L}(a_1 - L) \text{ and } T_2 = \frac{\lambda}{L}(a_2 - L)$$

$$\frac{T_1}{T_2} = \frac{a_1 - L}{a_2 - L}$$

or $$T_1 (a_2 - L) = T_2(a_1 - L)$$

or $$L(T_2 - T_1) = a_1T_2 - a_2T_1$$

or $$L = \frac{a_1T_2 - a_2T_1}{T_2 - T_1} = \frac{a_2T_1 - a_1T_2}{T_1 - T_2} \quad \text{...(i)}$$

Which gives the natural length of the string.

Also $$T_1 = \frac{\lambda}{L}(a_1 - L)$$

$$\frac{\lambda}{L} = \frac{T_1}{a_1 - L}$$

or $$\frac{\lambda}{L} = T_1 \Bigg/ \left[a_1 - \left(\frac{a_2T_1 - a_1T_2}{T_1 - T_2}\right)\right]$$

or $$\frac{\lambda}{L} = \frac{(T_1 - T_2)}{(a_1 - a_2)} \quad \text{...(ii)}$$

Now the required amount of work done

$$= \frac{1}{2}[0 + \frac{\lambda}{L}\{(a_1 + a_2) - L\}](a_1 + a_2 - L)$$

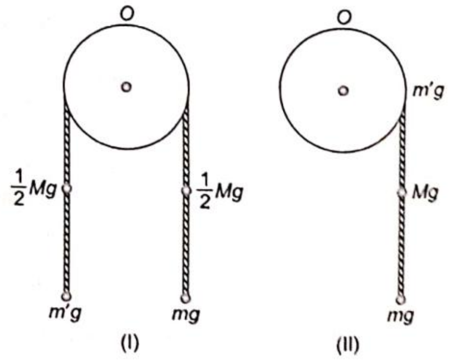
$$= \frac{1}{2}\left(\frac{T_1 - T_2}{a_1 - a_2}\right)(a_1 + a_2 - L)^2 \quad \text{[From Eq. (ii)]}$$

$$= \frac{1}{2}\left(\frac{T_1 - T_2}{a_1 - a_2}\right)\left[a_1 + a_2 - \frac{a_2T_1 - a_1T_2}{T_1 - T_2}\right]$$

$$= \frac{(a_1T_1 - a_2T_2)^2}{2(T_1 - T_2)(a_1 - a_2)}$$

**Example 67.** *A uniform string of mass M and length 2a is placed symmetrically over a smooth peg and has particles of masses m and m′ attached to its ends; show that when the strings run off the peg its velocity is*

$$\sqrt{\frac{M + 2(m - m')}{M + m + m'}ag}$$




**Solution**

**Case I** Depth of the centre of gravity of the system from the peg $O$.

$$= \frac{mga + m'ga + \frac{1}{2}Mg\frac{1}{2}a + \frac{1}{2}Mg\frac{1}{2}a}{mg + m'g + \frac{1}{2}Mg + \frac{1}{2}Mg}$$

$$= \frac{1}{2}a\frac{M + 2(m + m')}{M + m + m'} \quad \text{...(i)}$$

**Case II** Depth of the centre of gravity of the system when string runs off the peg

$$= \frac{mg2a + m'g0 + Mga}{Mg + m'g + mg} = a\frac{M + 2m}{M + m + m'} \quad \text{...(ii)}$$

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The initial velocity of the system was $O$ and the velocity of the system when the string runs off the peg is $v$.

Change in kinetic energy KE $= \frac{1}{2}(M + m + m')v^2 - \frac{1}{2}(M + m + m')0$

$$= \frac{1}{2}(M + m + m')v^2 \quad \text{...(iii)}$$

Also work done by the weight of the system

$$= (M + m + m')g \times \left[a\frac{M + 2m}{M + m + m'} - \frac{1}{2}a\frac{M + 2m + 2m'}{M + m + m'}\right]$$

$$= \frac{1}{2}ag[M + 2(m - m')]$$

Now, Change in kinetic energy = Work done

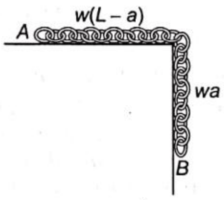
$$\therefore \quad \frac{1}{2}(M + m + m')v^2 = \frac{1}{2}ag[M + 2(m - m')]$$

$$v = \sqrt{\frac{M + 2(m - m')}{M + m + m'}ag}$$

**Example 68.** *A flexible but inextensible chain of length $L$ and weight $w$ is held on a smooth table with the length $(L - a)$ on the table and the length $a$ overhanging. Find the velocity with which the chain will leave the table when released.*

**Solution**

**Case I** The depth of centre of gravity of the chain below the table




$$= \frac{w(L-a)\times 0 + wa \times \frac{1}{2}a}{w(L-a) + wa} \quad (w = \text{weight per unit length})$$

$$= \frac{a^2}{2L} \quad \text{...(i)}$$

**Case II** The depth of centre of gravity when the chain is released

$$= \frac{1}{2}L \quad \text{...(ii)}$$

$\therefore$ Work done by the weight of the chain

$$= wL\left[\frac{L}{2} - \frac{a^2}{2L}\right] = \frac{w(L^2 - a^2)}{2} \quad \text{...(iii)}$$

Also initially the velocity of the chain was 0 and its velocity when it leaves the table is $v$.

The change in KE $= \frac{1}{2}\frac{wL}{g}v^2 - \frac{1}{2}\frac{wL}{g} \times 0$

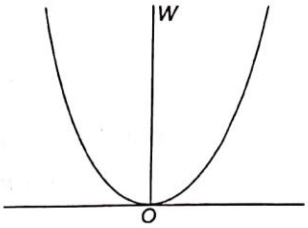
$$= \frac{1}{2}\frac{wL}{g}v^2$$

$\therefore$ Change in kinetic energy = Work done

$$\frac{1}{2}\frac{wL}{g}v^2 = \frac{w(L^2 - a^2)}{2}$$

$$v = \sqrt{g(L^2 - a^2)/L}$$

**Example 69.** *The dependence of the potential energy of an object on its position is given by the equation $W = ax^2$ (a parabola). What is the law by which the force acting on the object varies?*




**Solution** The work performed on an elementary segment of displacement is equal to the decrease in potential energy

$$dA = -dW$$

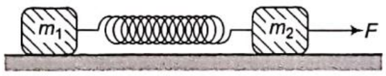
The same work can be represented as the product of force by displacement

$$dA = F_x dx.$$

Hence, $$F_x = -\frac{dW}{dx} = -2ax$$

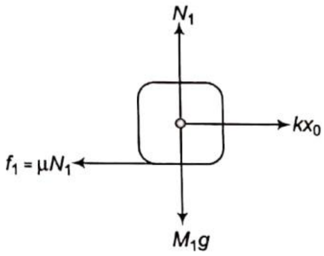
Forces known as quasielastic also obey this law.

**Example 70.** *Two blocks of masses $m_1$ and $m_2$ connected by an undeformed massless spring rests on a horizontal plane.*




*Find the minimum constant force F, that has to be applied in the horizontal direction to the block of mass $m_2$ so that the other block gets shifted, if $\mu$ be the coefficient of friction between blocks and surface.*

**Solution** As the force $F$ is applied on the block of mass $m_2$ it shifts the block towards right (if $F$ exceeds the friction force acting on the block). This process elongates the spring. So the restoring force generates in the spring and tends to move the block of mass $m_1$. If this restoring force exceeds the limiting frictional force, the block of mass $m_1$ moves.




For $m_1$

For vertical equilibrium

$$m_1 g = N_1$$

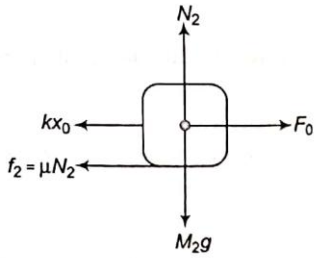
and for horizontal motion to impend

$$kx_0 \ge f_1$$

$\therefore$ $$f_1 = \mu N_1 \Rightarrow kx_0 \ge \mu m_1 g$$

but $x_0$ is minimum elongation

$$kx_0 = \mu m_1 g$$




For $m_2$

The force $F$ does the work in shifting by a distance $x_0$ against the frictional force

Besides, it also does work in elongating the spring by a length $x_0$.

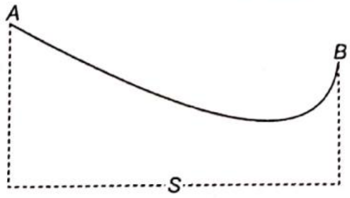
$\therefore$ $$Fx_0 = f_2 x_0 + \frac{1}{2}kx_0^2 = (\mu\, m_2 g)x_0 + \frac{1}{2}kx_0^2$$

$$F = \mu m_2 g + \frac{1}{2}kx_0$$

$$F = \mu m_2 g + \frac{1}{2}(um_1 g)$$

$$F = \mu g\left(m_2 + \frac{m_1}{2}\right)$$

**Example 71.** *A skier starts from rest at point A and slides down the hill without turning or breaking. The friction coefficient is $\mu$. When he stops at point B, his horizontal displacement is S. What is the height difference between points A and B. (The velocity of the skier is small so that the additional pressure on the snow due to the curvature can be neglected. Neglect also the friction of air and the dependence of $\mu$ on the velocity of the skier.)* [27th IPO 96]




**Solution** For a sufficiently shoil-horizontal displacement $\Delta S$ can be considered straight. If the corresponding length of path element is $\Delta L$, the friction force is given by $\mu mg \frac{\Delta S}{\Delta L}$ and the work done by the friction force equals force times displacement

$\Rightarrow$ $$\mu mg \frac{\Delta s}{\Delta L} \Delta L = \mu mg \Delta S$$

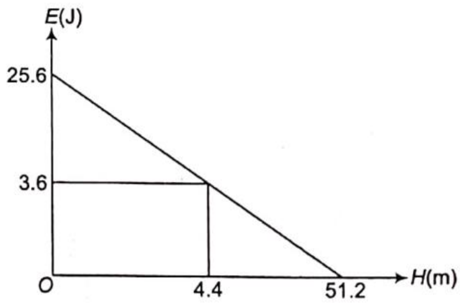
Adding up, we find that along the whole path the total work done by the friction force is $\mu mgs$. By energy conservation this must equals the decrease $mgh$ in potential energy of skier.

Hence, $h = \mu S$.

**Example 72.** *A stone of mass 50 g is thrown vertically upwards with a velocity of $12 \text{ ms}^{-1}$ from the edge of a cliff 44 m high.*

*Sketch a graph of the kinetic energy of the stone against height measured from the bottom of the cliff. Indicate the values of the kinetic energy at the top and bottom of the cliff and at its greatest height, specifying these heights on the graph. Ignore air resistance.* [British PO 1993]

**Solution**




$$\frac{1}{2}mv^2 = \frac{1}{2}mu^2 - mg(H - 44)$$

$$E = E_{44} - mg(H - 44)$$

At maximum height $E = 0$

$$H_{\max} = \frac{E_{44}}{mg} + 44 = 51.2 \text{ m.}$$

$$E_{44} = \frac{1}{2}(0.050)(12)^2 = 3.6 \text{ J}$$

At $H = 0$

$$E_0 = E_{44} + mg(44) = 25.6 \text{ J}$$

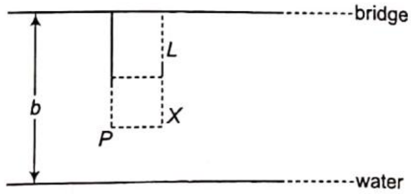
**Example 73.** *Bungee jumping is a sport in which a person jumps from a bridge using strong elastic rubber bands which have one end tied to the person's legs and the other end to the bridge so that the bands (eventually) stops the person's free fall. The length of the bands are adjusted so that the jumper just touches the surface of the water below the bridge.*

*Suppose that a jumper of mass 60 kg makes a successful jump from the Sydney Harbour Bridge from an initial height of 50 m above the water using bands with an unstretched length of 30 m.*

(*a*) *Determine the maximum speed of the jumper.*

(*b*) *What should be the length of the bands if the jumper has a mass of 90 kg. (Neglect the air resistance).*

[Australian PO-IV]




**Solution** At point $P$

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = mg(L+x) \qquad \text{...(i)}$$

At water $v = 0$

So, $$\frac{1}{2}k(b-l)^2 = mgb$$

$$k = \frac{2mgb}{(b-l)^2} \qquad \text{...(ii)}$$

Substituting values of $m$, $g$, $b$ and $l$ we get

$$k = 150 \text{ Nm}^{-1}$$

(a) Speed is maximum when the net force on the person is zero

$$mg - kx = 0, \ mg = kx \qquad \text{...(iii)}$$

$$x = \frac{mg}{k} = \frac{60 \times 10}{150} = 4 \text{ m}$$

From (i), (ii) and (iii) the maximum velocity is given by

$$\frac{1}{2}mv_m^2 = mg(l + x_m) - \frac{1}{2}kx_m^2$$

$$v_{\max} = 25.3 \text{ ms}^{-1}$$

(b) For stretching an elastic band the spring constant $k$ is inversely proportional to the length $l$ of the band.

$$k \propto \frac{1}{L}$$

Since, $$\frac{F}{A} = \frac{Ex}{L}$$

$$F = \frac{EAx}{L}$$

$$F = kx$$

So, $$k = \frac{EA}{L}$$

Let $k_1$, $L_1$ represents initial spring constant and length for $m_1 = 60$.
For $m_2 = 90$

$$k = k_2, \qquad L = L_2$$

$$\frac{k_2}{k_1} = \frac{L_1}{L_2} \Rightarrow k_2 = k_1\frac{L_1}{L_2}$$

$$k_2 = \frac{2m_2gb}{(b-L_2)^2}$$

$$\frac{k_1L_1}{L_2} = \frac{2m_2gb}{(b-L_2)^2}4$$

$$b^2 - 2bL_2 + L_2^2 = \frac{2m_2gbL_2}{K_1L_1}$$

$$L_2 = 26.8 \text{ m}$$

# Let us Practice

## Exercise (Level-1)

### Only One Correct Option

**1.** In the non-relativistic regime, if the momentum is increased by 100%, the percentage increase in kinetic energy is

(a) 100% (b) 200% **[NSEP 00-01]**
(c) 300% (d) 400%

**2.** A force of 200 N is required to push a car of mass 500 kg slowly at constant speed on a level road. If a force of 500 N is applied, the acceleration of the car (in $ms^{-2}$) will be

(a) zero (b) 0.2 (c) 0.6 (d) 1.0

**3.** In one dimensional motion, a 1 kg object experiences a force, which is a linear function of time $t$ *viz.* $F = 2t$ acting in the direction of motion. The work done by the force in first 4 s is **[NSEP 00-01]**

(a) 16 J (b) 32 J (c) 64 J (d) 128 J

**4.** A car accelerates from rest at a constant rate $\alpha$ for some time, after which it deaccelerates at a constant rate $\beta$ to come to rest. If the total time elapsed is $t$, the maximum velocity acquired by the car is given by **[NSEP 00-01 & 04-05]**

(a) $\left[\dfrac{\alpha\beta}{\alpha+\beta}\right]t$ (b) $\left[\dfrac{\alpha+\beta}{\alpha\beta}\right]t$

(d) $\left[\dfrac{\alpha^2+\beta^2}{\alpha\beta}\right]t$ (d) $\left[\dfrac{\alpha^2-\beta^2}{\alpha\beta}\right]t$

**5.** It is possible to project a particle with a given velocity in 2 possible ways so as to have the same range $R$. The product of corresponding times of flight is then proportional to

(a) $1/R$ (b) $R$ **[NESP 00-01]**
(c) $R^3$ (d) $1/R^2$

**6.** A given object takes $m$ times as much time to slide down a 45° rough incline as it takes to slide down a perfectly smooth 45° incline. The coefficient of kinetic friction between the object and the incline is given by **[NSEP 00-01]**

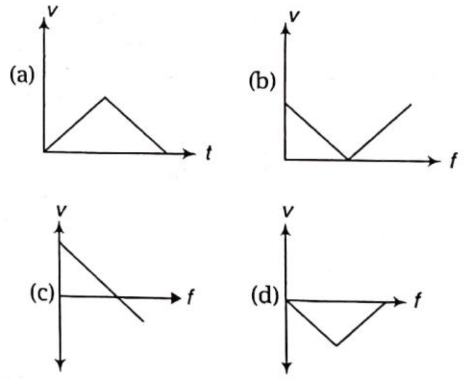
(a) $\mu_k = \dfrac{1}{1-m^2}$ (b) $\mu_k = 1 - \dfrac{1}{m^2}$

(c) $\mu_k = \sqrt{1-\dfrac{1}{m^2}}$ (d) $\mu_k = \sqrt{\dfrac{1}{1-m^2}}$

**7.** When a bucket containing water is rotated fast in a vertical circle of radius $R$, the water in the bucket doesn't spill provided that the bucket is

(a) whirled with a maximum speed of $\sqrt{2gR}$
(b) whirled around with a minimum speed of $\sqrt{(1/2)gR}$
(c) having a rpm of $n = \sqrt{900g/\pi^2R}$
(d) having a rpm of $n = \sqrt{3600g/\pi^2R}$

**8.** A ball is projected vertically upwards with a high velocity $v$. It comes back to ground in time $t$. Which $v$-$t$ graph shows the motion correctly? **[NSEP 00-01]**




**9.** Water is flowing in a river at 2.0 $ms^{-1}$. The river is 50 m wide and has average depth of 5.0 m. The power available from the current in the river is (density of water = 1000 $kg/m^3$) **[NSEP 00-01]**

(a) 0.5 MW (b) 1.0 MW
(c) 1.5 MW (d) 2.0 MW

**10.** Two forces of equal magnitude act at a point making an angle $\theta$ with each other. If the direction of one of the forces is reversed, the direction of the resultent will turn through **[NSEP 00-01]**

(a) 30° (b) 45°
(c) 60° (d) 90°

**11.** A particle of mass $m$ moves on the $x$-axis as follows; it starts from rest at $t = 0$ from the point $x = 0$, and comes to rest at $t = \phi$ at the point $x = 1$. No other information is available about its motion at intermediate times $(0<t<1)$. If $\alpha$ denotes the instantaneous acceleration of the particle, then **[NSEP 00-01]**

(a) $\alpha$ must remain positive for half the time
(b) $|\alpha|$ cannot exceed 2 at any point of time
(c) $|\alpha|$ must be $\geq 4$ at some point or points in its path
(d) $\alpha$ must change sign during the motion, but no other assertion can be made with the information given

**12.** Water is dripping from the top at a height $h$ from the ground at regular interval $t_1$. They reach the ground at interval $t_2$. Then $t_2$ is equal to

(a) $t_1$ (b) $t_1 + \sqrt{2h/g}$
(c) $t_1 - \sqrt{2h/g}$ (d) $t_1 + 2\sqrt{2h/g}$

**13.** A stone is dropped from the top of a tall cliff and $n$ seconds later another stone is thrown vertically downwards with a velocity $u$. Then the second stone over takes the first, below the top of the cliff at a distance given by **[NSEP 01-02]**

(a) $\frac{g}{2}\left[\frac{n\left(\frac{gn}{2} - u\right)}{gn - u}\right]^2$ (b) $\frac{g}{2}\left[\frac{n\left(gn - \frac{u}{2}\right)}{gn - u}\right]^2$

(c) $\frac{g}{2}\left[\frac{n\left(gn - \frac{u}{2}\right)}{gn - \frac{u}{2}}\right]^2$ (d) $\frac{g}{5}\left[\frac{gn - u}{gn - \frac{u}{2}}\right]^2$

**14.** An insect is crawling up on the concave surface of a fixed hemispherical bowl of radius $R$. If the coefficient of friction is $\frac{1}{3}$, then the height upto which the insect can crawl is nearly **[NSEP 01-02]**

(a) 5% of $R$ (b) 6% of $R$
(c) 6.5% of $R$ (d) 7.5% of $R$

**15.** A toy gun uses a spring of force constant $k$. When charge before being triggered in the upward direction, the spring is compressed by $x$. If the mass of the sheet is $m$, on being triggered, it will go up to a height of **[NSEP 01-02]**

(a) $kx^2 / 2mg$ (b) $kx^2 / mg$
(c) $k^2x^2 / mg$ (d) $x^2 / 3mg$

**16.** The speed of a train increases at a constant rate $\alpha$ from 0 to $v$ and then remains constant for an interval and finally decreases to zero, at a constant rate $\beta$. The total time taken to cover a total distance $l$ is given by **[NSEP 01-02]**

(a) $\left(\frac{l}{v}\right) + \left(\frac{v}{2}\right) \times \left(\frac{1}{\alpha} - \frac{1}{\beta}\right)$

(b) $\left(\frac{l}{v}\right) + \left(\frac{v}{2}\right) \times \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)$

(c) $\left(\frac{l}{v}\right) + \left(\frac{v}{2}\right)(\alpha\beta)$

(d) $\left(\frac{l}{v}\right)\alpha + \left(\frac{v}{2}\right)\beta$

**17.** A projectile is fired at an angle of 30° to the horizontal such that the vertical component of its initial velocity is 80 $ms^{-1}$. Its time of flight is $T$. Its velocity at the moment $t = T/4$, has a magnitude of nearly **[NSEP 01-02]**

(a) 200 $ms^{-1}$ (b) 160 $ms^{-1}$
(c) 150 $ms^{-1}$ (d) 140 $ms^{-1}$

**18.** Two statements are made in regard of three non-zero vectors $A$, $B$ and $C$ having different magnitudes.

I. Any two vectors can be combined to give a zero vector.
II. The three vectors can be combined to give a zero vector.

Then, **[NSEP 02-03]**

(a) Statement I and II are always true
(b) Statement I and II are always false
(c) Statement I is always wrong, but II may be true
(d) Statement I and II may be true

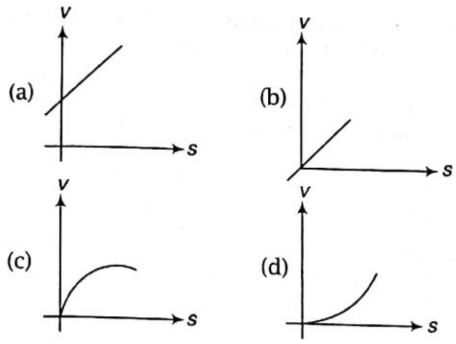
**19.** Which of the following combination of three different physical quantities $P$, $Q$, $R$ car never be a meaningful quantity? **[NSEP 02-03]**

(a) $PQ-R$ (b) $PQ/R$
(b) $(P-Q)/R$ (d) $(PR-Q)^2/QR$

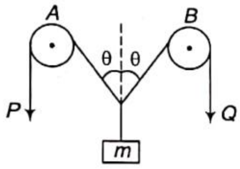
**20.** A chain of 5 links each of mass 0.1 kg is lifted vertically with a constant acceleration 1.2 $ms^{-2}$. The force of interaction between the top link and the one immediately below it is **[NSEP 02-03]**

(a) 5.5 N (b) 4.4 N
(c) 3.04 N (d) 7.6 N

**21.** A body starting from rest moves along a straight line with a constant acceleration. The variation of speed ($v$) with distance ($s$) is given by

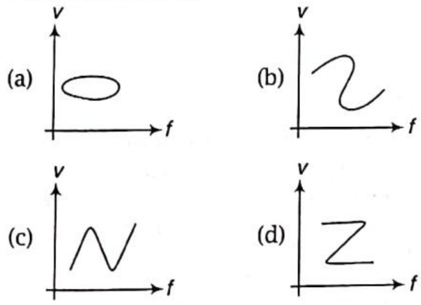


22. In the arrangement shown in the figure, the ends $P$ and $Q$ of an unstretchable string move downward with a uniform speed $u$. Pulleys $A$ and $B$ are fixed. The mass $m$ moves with a speed [NSEP 02-03]




(a) $2u\cos\theta$ (b) $u\cos\theta$
(c) $\dfrac{2u}{\cos\theta}$ (d) $\dfrac{u}{\cos\theta}$

23. Which of the following $v$-$t$ graphs shows a realistic situation?



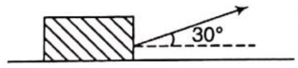
24. A particle moves rectilinearly. Its displacement $x$ as time $t$ is given by $x^2 = at^2 + b$. Its acceleration at time $t$ is proportional to [NSEP 03-04]

(a) $\dfrac{1}{x^3}$ (b) $\dfrac{1}{x} - \dfrac{1}{x^2}$
(c) $-t/x^2$ (d) None of these

25. An object is suspended from a spring balance in a lift. The reading is 240 N when the lift is at rest. If the spring balance reading now changes to 220 N, then the lift is moving [NSEP 03-04]]

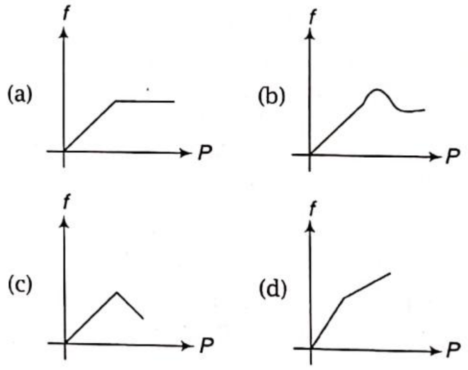
(a) downward with constant speed
(b) downward with decreasing speed
(c) downward with increasing speed
(d) upward with increasing speed

26. A box weighing 6 kg is being pulled with an acceleration 0.5 ms$^{-2}$ over a rough surface with the help of a string as shown in the figure. If the coefficient of kinetic friction involved is 0.1, then the tension in the string is [NSEP 03-04]




(a) 6 N
(b) 60 N
(c) more than 6 N but less than 60 N
(d) more than 60 N

27. An external horizontal force $P$ acts on a block placed on a rough horizontal surface. The force of friction between them is $f$. Which of the following graphs represents the reaction between $P$ and $f$ correctly? [NSEP 03-04]



28. An elastic spring of unstretched length $L$ and force constant $k$ is stretched by a small amount $x$. It is further stretched by another small length $y$. The work done in the second stretching is [NSEP 03-04]

(a) $\dfrac{1}{2}ky^2$ (b) $\dfrac{1}{2}k(x^2 + y^2)$
(c) $\dfrac{1}{2}k(x + y)^2$ (d) $\dfrac{1}{2}ky(2x + y)$

29. The resultant of two vectors $A$ and $B$ is perpendicular to vector $A$ and its magnitude is equal to half of the magnitude of vector $B$. Then, the angle between $A$ and $B$ is [NSEP 04-05]

(a) 30° (b) 45°
(c) 150° (d) 120°

**30.** A monkey of mass 20 kg rides on a 40 kg trolley moving at a constant speed of 8 m/s along a horizontal track. Frictional force can be neglected. If the monkey jumps vertically, with respect to the moving trolley, to catch the overhanging branch of tree, the speed of the trolley after the monkey has jumped off is **[NSEP 04-05]**

(a) 16 $ms^{-1}$ (b) 12$ms^{-1}$
(c) 8 $ms^{-1}$ (d) 6 $ms^{-1}$

**31.** If the momentum of a body increases by 20%, then the increase in its kinetic energy is **[NSEP 04-05]**

(a) 40% (b) 44%
(c) 48% (d) 56%

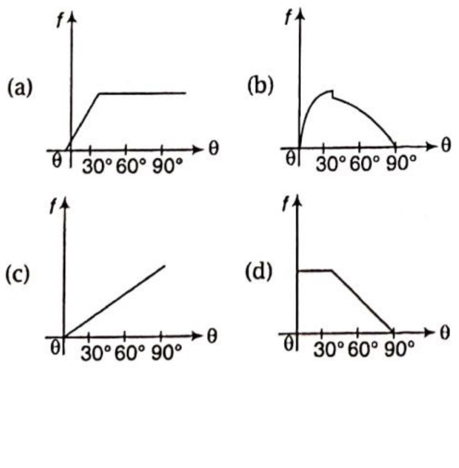
**32.** A car travelling at 36 km/h due North turns West in 5 s under constant acceleration and maintains the same speed. The acceleration of the car is **[NSEP 04-05]**

(a) $2\sqrt{2}$ $ms^{-2}$ South-West
(b) 10 $ms^{-2}$ North-East
(c) $2\sqrt{2}$ $ms^{-2}$ South-East
(d) 5 $ms^{-2}$ North-West

**33.** A lift whose cage is 3m high is moving up with an acceleration of 2 $ms^{-2}$. A piece of stone is dropped from the top of the cage of the lift when its velocity is 8 $ms^{-1}$. If $g = 10\,ms^{-2}$, then the stone will reach the floor of the lift after **[NSEP 04-05]**

(a) 0.7 s (b) 0.5 s
(c) 0.4 s (d) 0.3 s

**34.** A block is placed on a rough plane whose inclination to the horizontal ($\theta$) can be varied. The angle of repose is 30°. The graph that correctly indicates the variation of the frictional force ($f$) between the block and the plane with $\theta$ is



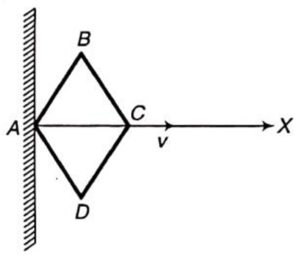

**35.** Two identical springs of force constant $k$ are connected (a) in series, (b) in parallel and support a mass $m$. The ratio of the period of oscillations of the series arrangement with that of the parallel arrangement is **[NSEP 04-05]**

(a) 1 : 1 (b) 1 : 2
(c) 1 : 4 (d) 2 : 1

**36.** The length of an elastic string is $a$ when the tension is 4 N and is $b$ when the tension is 5 N. The length when the tension is 9 N is **[NSEP 04-05]**

(a) $a + b$ (b) $4a + 5b$
(c) $5b - 4a$ (d) $9b - 9a$

**37.** Four rods each of length $l$ have been hinged to form a rhombus. Vertex $A$ is fixed to a rigid support, vertex $C$ being pulled to the right along $X$ axis with a uniform speed $v$ as shown. The speed at which vertex $B$ moves at the moment the rhombus takes the shape of a square is **[NSEP 04-05]**




(a) $v / 4$ (b) $v / 2$
(c) $v / \sqrt{2}$ (d) $v$

**38.** A particle moves with uniform acceleration and $v_1, v_2$ and $v_3$ represents the average velocities in three successive intervals of time $t_1, t_2$ and $t_3$ respectively, then **[NSEP 05-06]**

(a) $(v_1 + v_2)/(v_2 + v_3) = (t_1 - t_2)/(t_2 - t_3)$
(b) $(v_1 - v_2)/(v_2 - v_3) = (t_1 + t_2)/(t_2 + t_3)$
(c) $(v_1 - v_2)/(v_2 - v_3) = (t_1 - t_2)/(t_2 - t_3)$
(d) $(v_1 + v_2)/(v_2 + v_3) = (t_1 + t_2)/(t_2 + t_3)$

**39.** A particle is subjected to two equal forces along two different direction. If one of them is halved, the angle which the resultant makes with the other is also halved. The angle between the forces is **[NSEP 05-06]**

(a) 45° (b) 60°
(c) 90° (d) 120°

**40.** A body is projected at an angle of projection $\theta$, with kinetic energy $E$. Neglecting air friction, the kinetic energy at the highest point is **[NSEP 05-06]**

(a) zero (b) $E$
(c) $E\cos\theta$ (d) $E\cos^2\theta$

**41.** A particle moving along a straight line travels one third of the total distance with a speed of 3.0 m/s. The remaining distance is covered with a speed of 4.0 m/s for half the time and 5.0 m/s for the other half of the time. The average speed during the motion is **[NSEP 06-07]**

(a) 4.0 m/s (b) 6.0 m/s
(c) 3.8 m/s (d) 2.4 m/s

**42.** A boy throws a table tennis ball of mass 20 g upwards with a velocity of $u_0 = 10$ m/s at an angle $\theta_0$ with the vertical. The wind imparts a horizontal force of 0.08 N, so that the ball returns to the starting point. Then, the angle $\theta_0$ must be such that, $\tan\theta_0$ is **[NSEP 06-07]**

(a) 0.2 (b) 0.4
(c) 2.5 (d) 1.2

**43.** A cannon ball has a range $R$ on a horizontal plane, such that the corresponding possible maximum heights reached are $H_1$ and $H_2$. Then, the correct expression for $R$ is **[NSEP 06-07]**

(a) $(H_1 + H_2)/2$ (b) $(H_1H_2)^{1/2}$
(c) $2(H_1H_2)^{1/2}$ (d) $4(H_1H_2)^{1/2}$

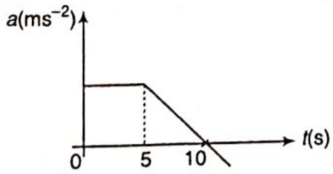
**44.** A windmill converts wind energy into electrical energy. If $v$ is the wind speed, electrical power output is proportional to

(a) $v$ (b) $v^2$
(c) $v^3$ (d) $v^4$

**45.** Vectors $a$ and $b$ include an angle $\theta$ between them. If $(a+b)$ and $(a-b)$ respectively subtend angles $\alpha$ and $\beta$ with $a$, then $(\tan\alpha + \tan\beta)$ is **[NSEP 08-09]**

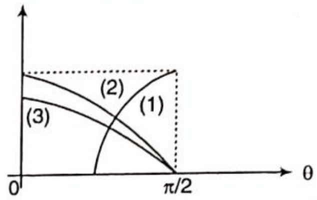
(a) $(ab\sin\theta)/(a^2 + b^2\cos^2\theta)$
(b) $(2ab\sin\theta)/(a^2 - b^2\cos^2\theta)$
(c) $(a^2\sin^2\theta)/(a^2 + b^2\cos^2\theta)$
(d) $(b^2\sin^2\theta)/(a^2 - b^2\cos^2\theta)$

**46.** Acceleration-time graph of particle moving along $X$-axis is as shown. The particle will have the velocity same as its initial velocity at **[NSEP 08-09]**




(a) 10 s (b) $(10 + \sqrt{3})$ s
(c) $(10 + 5\sqrt{3})$ s (d) $(10 + 2\sqrt{3})$ s

**47.** A block of mass $m$ is placed on an inclined plane with angle of inclination $\theta$. Let $N$, $f_L$ and $F$ respectively represent the normal reaction, limiting force of friction and the net force down the inclined plane. Let $\mu$ be the coefficient of friction. The dependence of $N$, $f_L$ and $F$ on $\theta$ is indicated by plotting graphs as shown below. Then curves (1), (2) and (3) respectively represent **[NSEP 08-09]**

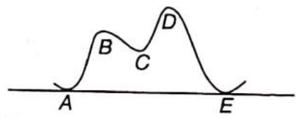



(a) $N$, $F$ and $f_L$ (b) $F$, $f_L$ and $N$
(c) $F$, $N$ and $f_L$ (d) $f_L$, $N$ and $F$

**48.** A block of weight 200 N is at rest on a rough inclined plane of inclination angle $\theta = 30°$. The inclined plane is at rest in the earth's inertial frame. Then the magnitude of the force the plane exerts on the block is **[INPhO 09]**

(a) $100\sqrt{3}$ N (b) 100 N
(c) 200 N (d) zero

**49.** A particle of mass 1 kg is taken along the path $ABCDE$ from $A$ to $E$ (see figure). The two "hills" are of heights 50 m and 100 m and the horizontal distance $AE$ is 20 m while the path length is 400 m. The coefficient of friction of the surface is 0.1. Take $g = 10$ ms$^{-2}$ and $\sqrt{3} = 1.73$. The minimum work on the mass required to accomplish this is **[INPhO 09]**



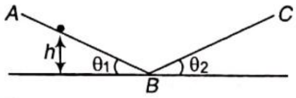

(a) 20 J (b) 173 J
(c) 400 J (d) 0 J

**50.** A uniform rope of length $L$ and mass $M$ partly lies on a horizontal table and partly hangs from the edge of the table. If $\mu$ is the coefficient of friction between the rope and the surface of the table (neglecting the friction at the edge), the maximum fraction of the length of the rope that overhangs from the edge of the table without sliding down is **[NSEP 09]**

(a) $\dfrac{1-\mu}{\mu}$ (b) $\dfrac{\mu}{\mu+1}$

(c) $1-\mu$ (d) $\dfrac{1}{\mu+1}$

**51.** A small glass bead of mass $m$ initially ar rest starts from a point at height $h$ above the horizontal and rolls down the inclined plane $AB$ as shown. Then, it rises along the inclined plane $BC$. Assuming no loss of energy, the time period of oscillation of the glass bead is [NSEP 09-10]



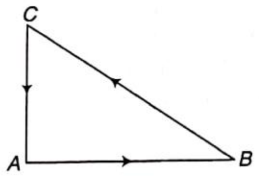

(a) $\sqrt{\dfrac{8h}{g}}(\sin\theta_1+\sin\theta_2)$

(b) $2\sqrt{\dfrac{14h}{5g}}\left(\dfrac{1}{\sin\theta_1}+\dfrac{1}{\sin\theta_2}\right)$

(c) $\sqrt{\dfrac{8h}{g}}\left(\dfrac{1}{\sin\theta_1}+\dfrac{1}{\sin\theta_2}\right)$

(d) $\sqrt{\dfrac{8h}{5g}\left(\dfrac{1}{\sin\theta_1}+\dfrac{1}{\sin\theta_2}\right)}$

**52.** Three forces start acting simultancy on a particle moving with velocity $\mathbf{v}$. These forces are represented in magnitude and direction by the three sides of a triangle $ABC$ as shown in the figure. The particle will now move with velocity




(a) greater than $\mathbf{v}$

(b) $|\mathbf{v}|$ in the direction of the largest force $BC$

(c) $\mathbf{v}$, remaining unchanged

(d) less than $\mathbf{v}$

**53.** A lift is moving downwards with an acceleration equal to acceleration due to gravity. A body of mass $m$ kept on the floor of the lift is pulled horizontally. If the coefficient of friction is $\mu$, then the frictional resistance offered by the body is

(a) $mg$ (b) $\mu mg$

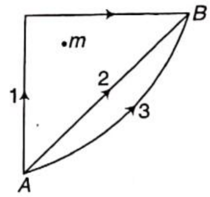
(c) $2\mu mg$ (d) zero

**54.** A particle of mass $m$ is moving in a circular path of constant radius $r$ such that its centripetal acceleration $a_c$ is varying with time $t$ as $a_c = k^2rt^2$, where $k$ is a constant. The power delivered to the particle by the force acting on it is

(a) $2\pi mk^2r^2t$ (b) $mk^2r^2t$

(c) $\dfrac{(mk^4r^2t^2)}{3}$ (d) zero

**55.** If $W_1$, $W_2$ and $W_3$ represent the work done in moving a particle from $A$ to $B$ along three different paths 1, 2 and 3 respectively (as shown) in the gravitational field of a point mass $m$, find the correct relation between $W_1$, $W_2$ and $W_3$.




(a) $W_1 > W_2 > W_3$ (b) $W_1 = W_2 = W_3$

(c) $W_1 < W_2 < W_3$ (d) $W_2 > W_1 > W_3$

## More than One Correct Options

**56.** Four forces act on a point object. The object will be in equilibrium, if [NSEP 00-01]

(a) all of them are in the same plane

(b) they are opposite to each other in pair

(c) the sum of $x, y$ and $z$ components is zero separately

(d) they form a closed figure of 4 sides

**57.** A particle of mass $m$ is projected with a velocity $v$ making an angle of 45° with the horizontal. The magnitude of the angular momentum of the projectile about the point of projection when the particle is at its maximum height $h$ is [NSEP 00-01]

(a) zero (b) $mv^3/(4\sqrt{2}g)$

(c) $mv^3/(\sqrt{2}g)$ (d) $m\sqrt{2gh^3}$

**58.** The displacement $x$ of a particle moving along a straight line, at time $t$ is given by $x = \dfrac{a}{b}(1-e^{-bt})$ where $a$ and $b$ are real and positive constants. Therefore [NSEP 01-02]

(a) at $t = 1/b$ the displacement of the particle is nearly $2a/3b$

(b) the velocity and acceleration of the particle at $t = 0$ are $a$ and $(-ab)$ respectively

(c) the particle cannot reach a point at a distance $x$ from its starting point if $x' > a/b$
(d) the particle will return to its starting point as $t \to \infty$

**59.** The distances covered by a freely falling body in its first, second, third,...$n^{th}$ seconds of its motion
(a) forms an arithmetic progression
(b) form a series corresponding to the squares of the first $n$ natural numbers
(c) do not form any well defined series
(d) forms a series corresponding to the difference of the squares of the successive natural numbers

**60.** Two springs $P$ and $Q$ have force constants $k_1$ and $k_2$ such that $k_1 > k_2$. The work done in stretching the two springs is **[NSEP 02-03]**
(a) more for spring $P$ than for spring $Q$ when they are stretched by the same force
(b) more for spring $P$ than for spring $Q$ when they are stretched by the same amount.
(c) more for spring $P$ than for spring $Q$ when they are stretched by the same amount but more for spring $Q$ when they are stretched by the same force
(d) more for spring $Q$ than for spring $P$ when they are stretched by the same amount but more for spring $P$ when they are stretched by the same force

**61.** For a curved track of radius $R$, banked at angle $\theta$ **[NSEP 02-03]**
(a) a vehicle moving with a speed $v_0 = \sqrt{Rg \tan\theta}$ is able to negotiate the curve without calling friction into play at all
(b) a vehicle moving with any speed $v > v_0$ is able to negotiate the curve with friction called into play
(c) a vehicle moving with any speed $v < v_0$ must also have the force of friction into play
(d) the minimum value of the angle of banking for a vehicle parked on the banked road can stay there without slipping, is given by $\theta = \tan^{-1}\mu_s$ ($\mu_s$ = coefficient of static friction)

**62.** Two particles are projected with the same initial velocity $v_0$, at two different angles of projection $\theta_1$ and $\theta_2$, such that their ranges are the same. The ratio of their heights reached is **[NSEP 03-04]**
(a) $\tan^2\theta_1$ (b) $\cot^2\theta_1$
(c) $\sin^2\theta_1 \operatorname{cosec}^2\theta_2$ (d) $\sec^2\theta_1 \cos^2\theta_2$

**63.** The motion of a body is given by the equation $dv(t)/dt = 6.0 - 3v(t)$ where $v(t)$ is the speed in m/s and $t$ is time in second. If the body was at rest at $t = 0$ **[NSEP 04-05]**
(a) the terminal speed is 2.0 m/s
(b) the magnitude of the initial acceleration is 6.0 m/s$^2$
(c) the speed varies with time as $v(t) = 2(1 - e^{-3t})$ m/s
(d) the speed is 1.0 m/s when the acceleration is half the initial value

**64.** A particle free to move along the $X$-axis has potential energy given by $U(x) = k[1 - \exp(-x^2)]$ for $-\infty \le x \le +\infty$, where $k$ is $a$ positive constant of appropriate dimensions. Then, **[NSEP 04-05]**
(a) at point away from the origin, the particle is in unstable equilibrium
(b) for any finite non zero value of $x$, there is a force directed away from the origin
(c) if its total mechanical energy is $k/2$, it has its minimum kinetic energy at the origin
(d) for small displacement from $x = 0$, the motion is simple harmonic

**65.** Let $v(t)$ be the velocity of a particle at time $t$. Then, **[NSEP 05-06]**
(a) $[dv(t)/dt]$ and $d|v(t)|/dt$ are always equal
(b) $dv|(t)|/dt$ and $d|v(t)|/dt$ may be equal
(c) $d|v(t)|/dt$ can be zero while $|dv(t)/dt|$ is not zero
(d) $d|v(t)|/dt \neq 0$ implies $|dv(t)/dt| \neq 0$

**66.** A monkey holds a light rope that passes over a smooth pulley. A bunch of bananas of equal mass as that of the monkey is attached to the other end of the rope. The monkey starts climbing the rope to get to the bananas. Then, **[NSEP 06-07]**
(a) the bananas also move up
(b) the bananas move downwards
(c) the distance between the monkey and the bananas decreases
(d) the distance between the monkey and the bananas remains constant

**67.** A particle of mass $m$ moves along a straight line under the action of a force $f$ varying with time as $f = f_0\left[1 - \left(\frac{t-T}{T}\right)^2\right]$, where $f_0$ and $T$ are positive constants. **[NSEP 09-10]**

(a) The speed of the particle after a time $2T$ is $\frac{4f_0T}{3m}$
(b) After time interval of $3T$, the particle starts moving backwards
(c) Between time instants 0 and $2T$, the acceleration first increases and then decreases
(d) The particle stops at $t = 3T$

**68.** Physical quantities $A$ and $B$ have the same dimensions. Then, **[NSEP 08-09]**
(a) $A \pm B$ must be a meaningful physical quantity
(b) $A \pm B$ may not be meaningful physical quantity
(c) $A/B$ must be a dimensionless quantity
(d) both must be either scalar or vector quantities

**69.** A particle moves in a straight line under the action of a constant force. Then, the graph of power developed by the force against **[NSEP 09-10]**
(a) time is a straight line
(b) time is a parabola
(c) displacement is a straight line
(d) displacement is a parabola

**70.** A particle is acted upon by a force of constant magnitude, which is always perpendicular to the velocity of the particle. The motion of the particle takes place in a plane. If follows that
(a) its velocity is constant
(b) its acceleration is constant
(c) its kinetic energy is constant
(d) it moves in a circular path

## Exercise (Level-2)

**1.** On a cricket field a rabbit is at the origin of coordinate system and a dog at the position (46 m, 28 m). The rabbit runs on the ground with a constant velocity $(7.5\,\mathbf{i} + 10\,\mathbf{j})$ m/s. The dog can run with a speed 5 m/s. If the dog starts to run immediately the rabbit starts, what is the minimum time in which the dog could catch the rabbit?

**2.** A particle starts from rest with zero initial acceleration. The acceleration increases uniformly with time. Find the time average of velocity upto a certain instant when the velocity becomes $v$.

**3.** A particle moves along a straight line such that its displacement $x$ from a fixed point on the line at time $t$ is given by

$$x^2 = at^2 + 2bt + c$$

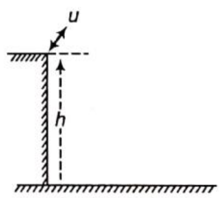
Find acceleration as a function of displacement $x$.

**4.** A particles moves along the $x$-axis. The velocity of the particle as a function of $x$ is given by

$$v^2 = 12x - 3x^3.$$

If $f$ be the instantaneous acceleration, express velocity as a function of $f$.

**5.** A particle is projected with an initial speed $u$ from a point at height $h$ above the horizontal plane as shown in the figure. Find the maximum range on the horizontal plane.

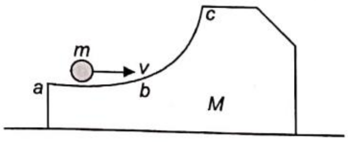



**6.** A disc revolves with a speed of $33\frac{1}{3}$ rev min$^{-1}$, and has a radius of 15 cm. Two coins are placed at 4 cm and 14 cm away from the centre of the disc. If the coefficient of friction between the coins on the disc is 0.15, which of the two coins will revolve with the disc?

**7.** A 70 kg man stands in contact against the wall of a cylindrical drum of radius 3m rotating about its vertical axis. The coefficient of friction between the wall and his clothing is 0.15. What is the minimum rotational speed of the cylinder to enable the man to remain stuck to the wall (without falling) when the floor is suddenly removed?

**8.** A body of mass $M = 9.8$ kg with a small disc of mass $m = 0.2$ kg placed on its horizontal surface $ab$, rests on a smooth horizontal plane, as shown in figure given below. The disc can move freely along the smooth groove $abc$ of mass $M$. To what height (relative to the initial position) will the disc rise after separating from the body of mass $M$, when

initial velocity $v = 5\ \text{ms}^{-1}$ is given to it the horizontal direction?




**9.** A ball of mass $m$ is dropped from a height $h$ on a platform fixed at the top of a vertical spring. The platform is depressed by a distance $x$. What is the spring constant $k$?

**10.** An aircraft is flying at a height of 3400 m above the ground. If the angle subtended at a ground observation point by the aircraft positions 10 s apart is 30°, what is the speed of the aircraft?

**11.** A cyclist is riding with a speed of 27 $\text{kmh}^{-1}$. As the approaches a circular turn on the road of radius 80 m, he applies brakes and reduces his speed at the constant rate 0.5 $\text{ms}^{-2}$. What is the magnitude and direction of the net acceleration of the cyclist on the circular turn?

**12.** A body travelling with uniform acceleration covers a distance 6 m in first 2s and 24 m in next 4 s of its journey. Find the velocity of the body of the end of 8 s from the beging.

**13.** A train of mass $m$ travelling with a constant power $P$ is observed to attain a speed $v_0$ at a certain instant of time. The air resistance to the train is proportional to its velocity $(kv)$. Find the time taken and the distance travelled by the train to double the speed. Assume that any other friction is negligible. **[INPhO 02]**

**14.** An amusement park ride consists of a large vertical cylinder that spins about its axis fast enough that any person inside is held up against the wall when the platform is removed. The coefficient of static friction between the wall and the person is 0.3 and radius of the cylinder is 5.0 m. Find the maximum period of revolution necessary to prevent the person from falling. How many revolutions per minute does the cylinder perform? **[INPhO 02]**

**15.** A particles starts from point $A$, moves along a straight line path with an acceleration given by $a = p - qx$ where $p$, $q$ are constants and $x$ is distance from point $A$. The particle stops at point $B$. Find the maximum velocity of the particle and the distance $AB$. **[INPhO 04]**

**16.** A particle of mass $m$ is projected horizontally with a velocity $v_0$ along the inner surface of a smooth hemispherical shell of radius $R$ at its rim. The particle moves along the inner surface of the sphere. Find the horizontal and the vertical components of velocity of the particle when it has descended a vertical distance $R/2$. **[INPhO 04]**

**17.** A horizontal force of magnitude $F$ is applied to a body of weight $mg$ resting on a frictionless inclined plane to prevent it from sliding down. The plane makes an angle of $\varphi$ with the horizontal. Then the find normal reaction acting on the body. **[NSEP 04-05]**

**18.** A car leaves point $A$ for point $B$ every 10 min. The distance between $A$ and $B$ is 60 km. The car travels at a speed of 60 km/h. Determine graphically only, the number of cars that a man driving from $B$ to $A$ will meet in route, if he starts from $B$ simultaneously with one of the cars leaving $A$. The car from $B$ travels with a speed of 60 km/h. **[INPhO 07]**

**19.** A mason lays four bricks to make an arch so that a portion of each brick protrudes over the one below. Determine the maximum lengths of the overhanging parts when the bricks are still in equilibrium without mortar. The length of each brick is 1. **[NIPhO 07]**

**20.** A block of mass $m$ is placed on a smooth horizontal surface. A force making an angle $\theta$ with the horizontal starts acting on the block. The magnitude of the force is constant but its direction with the horizontal changes as $\theta = a + bs$, where $a$ and $b$ are constants and $s$ is the distance covered by the block. If $|\mathbf{F}| = 2\,mb$, find the velocity of the block as a function of the angle $\theta$. **[NIPhO 07]**

# Solutions

## Exercise (Level-1)

**1.** (c) $KE = \dfrac{p^2}{2m}$

$\Rightarrow \quad KE' = \dfrac{p^2}{2m} = 4KE \qquad [\because p' = 2p]$

$\therefore \quad \Delta KE = 3KE = 300\%$ of KE

**2.** (c) As 200 N force is required to push the car with constant speed, opposing force must be 200 N, *i.e.*, $f = 200\,N$

$\therefore \quad a = \dfrac{F - f}{m}$

$= \dfrac{500 - 200}{500} = 0.6\ ms^{-2}$

**3.** (d) $F = 2t$

$\dfrac{dp}{dt} = 2t \Rightarrow p = t^2 \qquad (p = \text{momentum})$

Work done = change in KE $= \dfrac{p^2}{2m}$

$= \dfrac{t^4}{2}$

at $t = 4\,s$

$W = 128\,J$

**4.** (a) Let $v$ be the maximum velocity and $t_1$ be the time taken to acquire maximum velocity, then

$v = \alpha t_1 \Rightarrow t_1 = \dfrac{v}{\alpha} \qquad$ ...(i)

Also

$0 = v - \beta(t - t_1)$

$\Rightarrow \quad t - t_1 = \dfrac{v}{\beta} \qquad$ (ii)

Adding Eqs. (i) and (ii)

$t = \dfrac{v}{\alpha} + \dfrac{v}{\beta}$

$\Rightarrow \quad v = \dfrac{\alpha\beta}{\alpha + \beta} t$

**5.** (b) If $\theta_1$ and $\theta_2$ be two angles of projection, then

$\theta_1 + \theta_2 = \dfrac{\pi}{2}$

$\therefore \quad T_1 = \dfrac{2u \sin\theta_1}{g}$

and $\quad T_2 = \dfrac{2u\sin\theta_2}{g} = \dfrac{2u\cos\theta_1}{g}$

$\therefore \quad T_1T_2 = \dfrac{4u^2 \sin\theta_1 \cos\theta_1}{g^2} = \dfrac{2R}{g}$

$\therefore \quad T_1T_2 \propto R$

**6.** (b) $S = \dfrac{1}{2} a_1 t_1^2 = \dfrac{1}{2} a_2 t_2^2$

$\Rightarrow \quad \dfrac{a_2}{a_1} = \left(\dfrac{t_1}{t_2}\right)^2$

$\Rightarrow \quad \dfrac{g(\sin 45° - u_k \cos 45°)}{g \sin 45°} = m^2$

$\Rightarrow \quad u_k = 1 - \dfrac{1}{m^2}$

**7.** (c) Angular velocity is minimum at the highest point.

At the highest point

$m\omega^2 R = mg$

$\omega = \sqrt{\dfrac{g}{R}}$

$\text{rpm} = \dfrac{60}{2\pi}\omega = \sqrt{\dfrac{900g}{\pi^2 R}}$

**8.** (c) $v = u - gt$

**9.** (b) $P = \dfrac{dK}{dt} = \dfrac{1}{2}\dfrac{dm}{dt}v^2$ ($K$ = kinetic energy)

$= \dfrac{1}{2}\rho A V^3$

$= \dfrac{1}{2} \times 10^3 \times (5 \times 50) \times (2)^3 = 10^6\ W = 1.0\ MW$

**10.** (d) $\mathbf{R}_1 = \mathbf{F}_1 + \mathbf{F}_2$, $\mathbf{R}_2 = \mathbf{F}_1 - \mathbf{F}_2$

If $\alpha$ be the angle between $\mathbf{R}_1$ and $\mathbf{R}_2$, then

$\cos\alpha = \dfrac{\mathbf{R}_1 \cdot \mathbf{R}_2}{R_1 R_2}$

$= \dfrac{F_1^2 - F_2^2}{F_1^2 + F_2^2}$

but $\quad F_1 = F_2$

$\Rightarrow \quad \cos\alpha = 0$ and $\alpha = 90°$

**Alternate Method**

Two equal forces may represent adjacent sides of a rhombus. Then $\mathbf{R}_1$ and $\mathbf{R}_2$ represent two diagonals of the rhombus and diagonals of rhombus are always perpendicular to each other.

**11.** (d) As the total change in velocity is zero. Acceleration must be positive for some time and negative for remaining time.

**12.** (a) Time taken by each drop of water is equal.

**13.** (a) Let the second stone crosses first stone at a distance $x$ from the top, then

For first stone

$$x = \frac{1}{2} g(t + n)^2 \quad \text{...(i)}$$

and for second stone

$$x = ut + \frac{1}{2} gt^2 \quad \text{...(ii)}$$

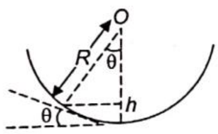
From Eqs. (i) and (ii)

$$\frac{1}{2} g(t + n)^2 = ut + \frac{1}{2} gt^2$$

$$\Rightarrow \quad t = \frac{gn^2}{2(u - gn)}$$

$$x = \frac{g}{2}\left[\frac{n\left(\frac{gn}{2} - u\right)}{gn - u}\right]^2$$

**14.** (a) Let $\theta$ be the maximum angle of radius vector of insect, made with the vertical, then $\theta$ must be the angle of repose, *i.e.*,




$$\tan\theta = \mu = \frac{1}{3}$$

$$\Rightarrow \cos\theta = \frac{1}{\sqrt{1 + \tan^2\theta}} = \frac{3}{\sqrt{10}}$$

$$\Rightarrow \quad h = R(1 - \cos\theta) = \frac{\sqrt{10} - 3}{\sqrt{10}} R = 5\% \text{ of } R$$

**15.** (a) By conservation of energy

$$mgh = \frac{1}{2} kx^2$$

$$\Rightarrow \quad h = \frac{kx^2}{2mg}$$

**16.** (b) Let the train accelerates for time $t_1$, moves with uniform speed for time $t_2$ and retards for time $t_3$.

Let $S_1$, $S_2$ and $S_3$ respectively be the distance travelled in these three intervals, then

$$S_1 = \frac{1}{2} \alpha t_1^2, \quad \text{...(i)}$$

$$S_2 = v\, t_2 \quad \text{...(ii)}$$

and $\quad S_3 = vt_3 - \frac{1}{2} \beta t_3^2$

Also $\quad v = \alpha t_1 = \beta\, t_3$

$$\therefore \quad S_1 = \frac{1}{2} v\, t_1$$

$$S_2 = v\, t_2$$

$$S_3 = \frac{1}{2} v\, t_3$$

$$\therefore \quad t = t_1 + t_2 + t_3 = \frac{2S_1}{v} + \frac{S_2}{v} + \frac{2S_3}{v}$$

$$= \frac{l}{v} + \frac{v}{2}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)$$

$$\left[\because S_1 = \frac{v^2}{2\alpha} \text{ and } S_2 = \frac{v^2}{2\beta}\right]$$

**17.** (d) At $t = \frac{T}{4}$, $u = \frac{80}{\sin\theta} = 160$

$$v_x = u\cos\theta = 80\sqrt{3} \text{ ms}^{-1}$$

$$v_y = u\sin\theta - gt = 80 - 10\frac{T}{4}$$

$$= 80 - 40 = 40 \text{ ms}^{-1} \quad (\because T = 16 \text{ s})$$

$$\therefore \quad v = \sqrt{v_x^2 + v_y^2} \approx 140 \text{ ms}^{-1}$$

**18.** (c)

**19.** (c) Two different physical quantities can never be added or subtracted.

**20.** (b) $T - 4mg = 4ma$

$$T = 4m(g + a)$$

$$= 4 \times 0.1 \times 11$$

$$= 4.4 \text{ N}$$


**21.** (d) $v^2 \propto s$

**22.** (d) Let $v$ be the speed of the block, then

$$u = v\cos\theta$$

$$\Rightarrow \quad v = \frac{u}{\cos\theta}$$

**23.** (c) A body cannot have two velocities at a given instant of time.

**24.** (a) Given

$$x^2 = at^2 + b$$

$$\Rightarrow \quad 2x\frac{dx}{dt} = 2at$$

$$\Rightarrow \quad x\,v = at$$

$$\Rightarrow \quad x\frac{dv}{dt} + v\frac{dx}{dt} = a$$

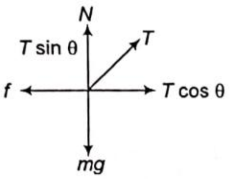
$\therefore$ Acceleration,

$$\alpha = \frac{a - v^2}{x}$$

$$= \frac{a(x^2 - at^2)}{x^3} = \frac{ab}{x^3}$$

**25.** (c) As the reading is decreasing, lift must be acceleration downwards.

**26.** (c) FBD of box is shown below




From the FBD

$$T\cos\theta - f = ma \quad \text{...(i)}$$

$$N + T\sin\theta = mg$$

$$\Rightarrow \quad N = mg - T\sin\theta \quad \text{...(ii)}$$

and $\quad f = \mu N$

$$T\cos\theta - ma = \mu(mg - T\sin\theta)$$

$$\Rightarrow \quad T = \frac{m(\mu g + a)}{\cos\theta + \mu\sin\theta} = \frac{18}{\sqrt{3} + 0.1}$$

$$\therefore \quad 6\text{N} < T < 60\text{N}.$$

**27.** (b)

**28.** (d) $W = \frac{1}{2}k(x_2^2 - x_1^2)$

$$= \frac{1}{2}k[(x + y)^2 - x^2]$$

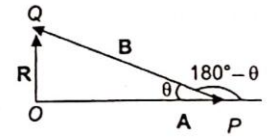
$$= \frac{1}{2}ky(2x + y)$$

**29.** (c) Situation is shown is figure

As $\quad OQ = \frac{1}{2}PQ$

$$\theta = 30^\circ$$

$$\therefore \quad 180^\circ - \theta = 150^\circ$$



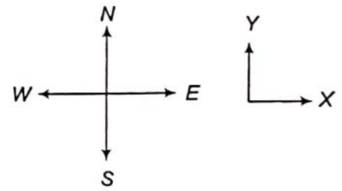

**30.** (c) Horizontal momentum remains same.

**31.** (b) $K = \frac{p^2}{2m}$, $K' = \frac{p^2}{2m} = 1.44\,K$

$[p' = 1.2p]$

$\Delta K = K' - K = 0.44\,K = 44\%$ of $K$

**32.** (a) $\mathbf{v}_1 = 36\hat{\mathbf{j}}$ km/h $= 10\hat{\mathbf{j}}$ ms$^{-1}$




$$\mathbf{v}_2 = -10\,\hat{\mathbf{i}}\,\text{ms}^{-1}$$

$$\therefore \mathbf{v}_2 - \mathbf{v}_1 = -10\,\hat{\mathbf{i}} - 10\,\hat{\mathbf{j}}$$

$$\mathbf{a} = \frac{\mathbf{v}_2 - \mathbf{v}_1}{t} = -2\hat{\mathbf{i}} - 2\,\hat{\mathbf{j}}$$

$$\mathbf{a} = 2\sqrt{2}\ \text{ms}^{-2}\ \text{south-west}$$

**33.** (a) Initial velocity of stone wrt lift, $u = 0$

Acceleration of stone wrt lift

$$= 10 + 2 = 12\ \text{ms}^{-2}$$

$\therefore$ Using $S = ut + \frac{1}{2}at^2$

$$3 = \frac{1}{2} \times 12t^2$$

$$\Rightarrow \quad t = 0.7\ \text{s}$$

**34.** (b) $\quad f = mg\sin\theta$ for $\theta \le 30^\circ$

$\quad u_k mg\cos\theta$ for $\theta > 30^\circ$

**35.** (d) For series combination

$$k_s = \frac{k_1 k_2}{k_1 + k_2} = \frac{k}{2}$$

$$\therefore \quad T_s = 2\pi\sqrt{\frac{m(k_1 + k_2)}{k_1 k_2}}$$

$$= 2\pi\sqrt{\frac{2m}{k}}$$

For paralled combination

$$k_p = k_1 + k_2$$

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$$T_p = 2\pi\sqrt{\frac{m}{k_1 + k_2}} = 2\pi\sqrt{\frac{m}{2k}}$$

$\therefore \quad \frac{T_s}{T_p} = 2$

**36.** (c) Let $L$ be the natural length of the string and $k$ be its force constant, then

$u = k(a - L)$ ...(i)

and $s = k(b - L)$ ...(ii)

On solving

$$L = 5a - 4b \text{ and } k = \frac{1}{b-a}$$

Let $c$ be the length of string when tension is 9 N, then

$$k(c - L) = 9$$

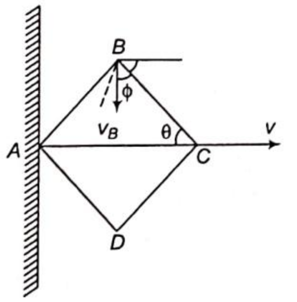
$$\Rightarrow \quad c = \frac{9}{k} + L$$

$$\Rightarrow \quad c = 9(b - a) + 5a - 4b = 5b - 4a$$

**37.** (c) Let $v_B$ be the velocity of point $B$ at any instant in the direction as shown in figure.

$v_B \cos\phi = v\cos\theta$ ...(i)

$v_B \cos(\phi + \theta) = \frac{v}{2}$ ...(ii)




when the rhombus will take the shape of the square,

$$\theta = 45^\circ$$

$\therefore \quad \phi = 0^\circ$ and $v_B = \frac{v}{\sqrt{2}}$

**38.** (b) Let the particle starts from point $O$, reaches point $A$ in time $t_1$, travels from $A$ to $B$ in time $t_2$ and from $B$ to $C$ in time $t_3$.

Let $a$ be the acceleration of the particle and $v_O, v_A, v_B$ and $v_C$ velocities of particle at $O$, $A$, $B$ and $C$ respectively. Then

$v_A = v_O + a t_1$ ...(i)

$v_B = v_A + at_2$ ...(ii)

$v_C = v_B + at_3$ ...(iii)

Also $\quad v_1 = \frac{v_O + v_A}{2}$,

$v_2 = \frac{v_A + v_B}{2}$

and $\quad v_3 = \frac{v_B + v_C}{2}$,

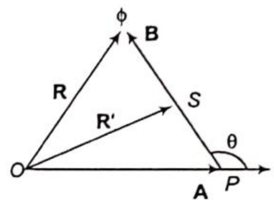
On solving

$$(v_1 - v_2) : (v_2 - v_3) = (t_1 + t_2) : (t_2 + t_3)$$

**39.** (d) The situation is shown in figure below **R′** vector is median of triangle at **B** draw $O$.

As **R′** bisects $\angle QOP$, triangle $OPQ$ must be equilateral

$\therefore \quad \theta = 120^\circ$




**40.** (d) Let $u$ be the initial velocity, then

$$E = \frac{1}{2}mu^2$$

At the highest point

$$v = u\cos\theta$$

$$\therefore \quad E' = \frac{1}{2}mv^2 = E\cos^2\theta$$

**41.** (c) Let $v'$ be the velocity for last two-third distance, then

$$v' = \frac{v_2 + v_3}{2}$$

$\therefore$ Average velocity for entire distance

$$\langle v \rangle = \frac{s}{t_1 + t_2} + \frac{s}{\frac{s}{3v_1} + \frac{2s}{3\left(\frac{v_1 + v_2}{2}\right)}}$$

$$= \frac{3}{\frac{1}{v_1} + \frac{4}{v_1 + v_2}} = \frac{3}{\frac{1}{3} + \frac{4}{9}}$$

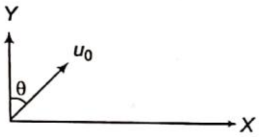
$$= \frac{27}{7} = 3.8\,\text{ms}^{-1}$$

**42.** (b) We have,

$$u_x = u_0 \sin\theta_0,$$

$$u_y = u_0 \cos\theta_0$$

$$a_x = -\frac{0.08}{0.02} = -4\ \text{ms}^{-2},\ a_y = -g = -10\ \text{ms}^{-2}$$




$\therefore$ $$x = u_x t + \frac{1}{2}a_x t^2$$

$$= u_0 \sin\theta_0\, t - 2t^2 \quad \text{...(i)}$$

and $$y = u_y t + \frac{1}{2}a_y t^2$$

$$= u_0 \cos\theta_0\, t - 5t^2 \quad \text{...(ii)}$$

If the ball reaches the point of projection again, then

$$x = 0, y = 0$$

On solving

$$\tan\theta_0 = 0.4$$

**43.** (d) $R = \dfrac{u^2 \sin 2\theta}{g}$

$$H_1 = \frac{u^2\sin^2\theta}{2g},$$

$$H_2 = \frac{u^2\cos^2\theta}{2g}$$

$$H_1H_2 = \left(\frac{u^2\sin\theta\cos\theta}{2g}\right)^2$$

$$= \left(\frac{u^2\sin 2\theta}{4g}\right)^2$$

$\Rightarrow$ $$H_1H_2 = \left(\frac{R}{4}\right)^2$$

$\Rightarrow$ $$R = 4\sqrt{H_1H_2}$$

**44.** (c) $P = \dfrac{dK}{dt} = \dfrac{1}{2}\dfrac{dm}{dt}v^2$

[$K$ = kinetic energy]

$$P = \frac{1}{2}(\rho Av)v^2 = \frac{1}{2}\rho A v^3$$

$\therefore$ $$P \propto v^3$$

**45.** (b) $\tan\alpha = \dfrac{b\sin\theta}{a + b\cos\theta}$

and $$\tan\beta = \frac{b\sin\theta}{a - b\cos\theta}$$

$$\tan\alpha + \tan\beta = \frac{b\sin\theta}{a+b\cos\theta} + \frac{b\sin\theta}{a-b\cos\theta}$$

$$= \frac{b\sin\theta(a - b\cos\theta + a + b\cos\theta)}{a^2 - b^2\cos^2\theta}$$

$$= \frac{2ab\sin\theta}{a^2 - b^2\cos^2\theta}$$

**46.** (c) Change in velocity is zero when area under $a$ - $t$ graph is zero *i.e.*,

$$\frac{1}{2}(10+5)a_0 + \frac{1}{2}(t-10)a = 0 \quad \text{...(i)}$$

Also $$\frac{a-0}{t-10} = \frac{0-a_0}{10-5}$$

$\Rightarrow$ $$a = -\frac{a_0}{5}(t-10) \quad \text{...(ii)}$$

From Eqs. (i) and (ii)

$$15a_0 - \frac{a_0}{5}(t-10)^2 = 0$$

$$75 - t^2 + 20t - 100 = 0$$

$$t^2 - 20t + 25 = 0$$

$\Rightarrow$ $$t = \frac{20 \pm \sqrt{400-100}}{2}$$

$$= (10 + 5\sqrt{3})\ \text{s} \quad (\because t > 10)$$

**47.** (c) $N = mg\cos\theta$

$$f_L = \mu N$$

$$F = mg\cos\theta - f$$

**48.** (c) As block is at rest net force acting on the block must be zero.

**49.** (a) $W = \mu\, mgL = 0.1 \times 1 \times 10 \times 20 = 20\text{J}$

(Work done against friction depend only on the horizontal distance.)

**50.** (b) For equilibrium,

Weight of hanging portion = Frictional force on portion lying on table

$$\frac{m}{L}lg = \mu\frac{m}{L}(L-l)g$$

$\Rightarrow$ $$\frac{l}{L} = \frac{\mu}{\mu+1}$$

**51.** (c) Time taken by the bead to reach $B$,

$$t_1 = \sqrt{\frac{2h}{2\sin^2\theta_1}}$$

Time taken by the bead to reach maximum height on $BC$,

$$t_2 = \sqrt{\frac{2h}{g \sin^2 \theta_2}}$$

$\therefore$ Time period, $T = 2(t_1 + t_2)$

$$= \sqrt{\frac{8h}{g}} \left( \frac{1}{\sin \theta_1} + \frac{1}{\sin \theta_2} \right)$$

**52.** (c) As three forces are forming a closed loop, net force is zero, hence velocity will remain unchanged.

**53.** (d) $N = m(g - a) = 0$

$$f = \mu N = 0$$

Hence no force is required to drag the block.

**54.** (b) $a_c = \frac{v^2}{r} = k^2 r t^2$

Kinetic energy,

$$K = \frac{1}{2} mv^2 = \frac{1}{2} mk^2 r^2 t^2$$

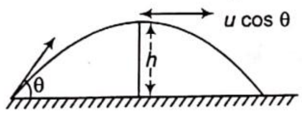
$$P = \frac{dK}{dt} = mk^2 r^2 t$$

**55.** (b) Gravitational force is a conservative force and work done by conservative force does not depend on path.

**56.** (c, d)

**57.** (b, d) At the highest point,

Angular momentum about point of projection




$$L = mv \cos \theta \, h = \frac{1}{\sqrt{2}} mvh$$

But $\quad h = \frac{v^2 \sin^2 \theta}{2g} = \frac{v^2}{4g}$

$\therefore \quad L = \frac{mv^3}{4\sqrt{2}g}$

Also $\quad v = \sqrt{4gh}$

$$L = \frac{m(4gh)^{3/2}}{4\sqrt{2}g} = m\sqrt{2gh^3}$$

**58.** (a, b, c, d) At $t = \frac{1}{b}, x = \frac{a}{b}(1 - e^{-1})$

$$= 0.63 \frac{a}{b} \approx \frac{2a}{3b}$$

Velocity

$$v = \frac{dx}{dt} = a e^{-bt}$$

Acceleration $A = \frac{dv}{dt} = -ab\, e^{-bt}$

At $\quad t = 0$

$$v = a, A = -ab$$

Also $\quad x_{max} = \frac{a}{b}$

At $\quad t = \infty, x = 0$

**59.** (a, d)

$$D_n = \frac{a}{2}[n^2 - (n-1)^2]$$

$$= \frac{a}{2}(2n - 1)$$

**60.** (b, c) $W = \frac{1}{2} kx^2 = \frac{F^2}{2k}$

**61.** (a, b, c, d)

**62.** (a, b, c, d) For same range

$$\theta_1 + \theta_2 = \frac{\pi}{2}$$

$$H_1 = \frac{u^2 \sin^2 \theta_1}{2g},$$

$$H_2 = \frac{u^2 \sin^2 \theta_2}{2g},$$

$$\frac{H_1}{H_2} = \frac{\sin^2 \theta_1}{\sin^2 \theta_2} = \sin^2 \theta_1 \operatorname{cosec}^2 \theta_2$$

Also $\quad \frac{H_1}{H_2} = \frac{\cos^2 \theta_2}{\cos^2 \theta_1} = \sec^2 \theta_1 \cos^2 \theta_2,$

$$\frac{H_1}{H_2} = \frac{\sin^2 \theta_1}{\sin^2 \theta_2} = \frac{\sin^2 \theta_1}{\cos^2 \theta_1} = \tan^2 \theta_1$$

and $\quad \frac{H_1}{H_2} = \cot^2 \theta_1$

**63.** (a b, c) $\frac{dv}{dt} = 6 - 3v \Rightarrow \frac{dv}{2 - v} = 3\, dt$

$\Rightarrow \quad \int_0^v \frac{dv}{2 - v} = 3\int_0^t dt$

$\Rightarrow \quad -\ln \frac{2 - v}{2} = 3t$

$$v = 2(1 - e^{-3t})$$

At $\quad t = 0, a = 6 \text{ ms}^{-2}$

$$v_{max} = 2 \text{ ms}^{-1}$$

$$a = \frac{dv}{dt} = 6e^{-2t}$$

Acceleration is half the maximum value i.e., $3\ \text{ms}^{-2}$.

At $t = \frac{1}{2}\ln 2$ s

At this instant,

$$v = 3(1 - e^{-\ln 2}) = \frac{3}{2}\ \text{ms}^{-1}$$

**64.** (a, b, d)

$$U(x) = k[1 - \exp(-x^2)] \quad \text{...(i)}$$

$$F = -\frac{dU}{dx} = -2kx[\exp(-x^2)] \quad \text{...(ii)}$$

At $\quad t = \pm\infty$

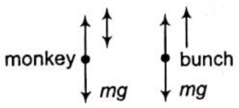
$U(x) = \text{maximum}$

Hence, particle is in unstable equilibrium.

From Eq. (ii), force is directed away from origin.

**65.** (b, c)

**66.** (a, d) FBD of monkey and bunch of bananas are shown below




As equal force is acting on both and both have equal mass, hence both will have same acceleration and in same direction.

**67.** (a, b, c, d)

$$f = f_0\left[1 - \left(\frac{t+T}{T}\right)^2\right]$$

$$f = \frac{f_0}{T^2}[T^2 - T^2 - t^2 + 2Tt]$$

$$f = \frac{f_0}{T^2}[2Tt - t^2]$$

$$m\frac{dv}{dt} = \frac{f_0}{T^2}[2Tt - t^2]$$

$$v = \frac{f_0}{T^2}\left[Tt^2 - \frac{t^3}{3}\right]$$

Now put values of $t$ and find $u$ and $a$.

**68.** (b, c) Two different quantities may have same dimensions.

**69.** (a, d) $P = Fv$

but $\quad v = u + at = \sqrt{u^2 + 2as}$

$\therefore \quad P = F(u + at) = F\sqrt{u^2 + 2as}$

**70.** (a, d) If the force has constant magnitude and acts perpendicular to velocity, it provides centripetal force.

## Exercise (Level-2)

**1.** Let the velocity of dog,

$$\mathbf{v}_d = a\mathbf{i} + b\mathbf{j}$$

where $\quad a^2 + b^2 = 5^2 = 25$

$$\mathbf{v}_{dr} = (a - 7.5)\mathbf{i} + (b - 10)\mathbf{j}$$

$$x = v_{dr_x} t = (a - 7.5)t$$

$$a = \frac{46}{t} + 7.5 \quad \text{...(i)}$$

Similarly,

$$y = v_{dr_y}\ t = (b - 10)t$$

$$b = \frac{28}{t} + 10 \quad \text{...(ii)}$$

But $\quad a^2 + b^2 = 25$

$$\Rightarrow \quad \left(\frac{46}{t} + 7.5\right)^2 + \left(\frac{28}{t} + 10\right)^2 = 25$$

On solving,

$$t = 4\ \text{s},\ 5.25\ \text{s}$$

$\therefore$ Minimum time = 4 s

**2.** Given $a = kt$ where $k$ = constant

$$\frac{dv}{dt} = kt$$

$$v = \frac{1}{2}kt^2$$

$$s = \frac{1}{6}kt^3$$

$$\text{Average velocity} = \frac{s}{t} = \frac{1}{6}kt^2 = \frac{v}{3}$$

**3.** Given,

$$x^2 = at^2 + 2bt + c \quad \text{...(i)}$$

$$2x\frac{dx}{dt} = 2at + 2b = 2(at + b)$$

$$\Rightarrow \quad x\frac{dx}{dt} = at + b$$

$$\Rightarrow \quad v^2 + x\frac{dv}{dt} = a$$

$$\Rightarrow \quad \frac{dv}{dt} = \frac{a - v^2}{x}$$

$$\Rightarrow \quad \frac{dv}{dt} = \frac{1}{x}\left[a - \left(\frac{at + b}{x}\right)^2\right] \quad \text{...(ii)}$$

From Eq. (i)

$$t = \frac{-b + \sqrt{b^2 - a(c - x^2)}}{a}$$

$$\Rightarrow \quad (b + at) = \sqrt{b^2 - a(c - x^2)}$$

Substituting this value in Eq. (ii) we get

$$\frac{dv}{dt} = \frac{ac - b^2}{x^3}$$

**4.** $v^2 = 12x - 3x^2$ ...(i)

$$\Rightarrow \quad 2v\frac{dv}{dx} = (12 - 9x^2)v$$

$$\Rightarrow \quad 2f = 12 - 9x^2$$

$$\Rightarrow \quad x^2 = \frac{12 - 2f}{9} \quad \text{...(ii)}$$

From Eq.(i)

$$v^4 = (12x - 3x^3)^4 = 9x^2(4 - x^2)$$

$$= (12 - 2f)\left[4 - \left(\frac{12 - 2f}{9}\right)\right]$$

$$\Rightarrow \quad v = \frac{1}{3}[8(6 - f)(12 + f)^2]^{1/4}$$

**5.** Let $\theta$ be the angle of projection, then

$$R = (u\cos\theta)t \quad \text{...(i)}$$

and
$$-n + \frac{1}{2}gt^2 = (u\sin\theta)t \quad \text{...(ii)}$$

From Eqs.(i) and (ii)

$$R^2 + \left(\frac{1}{2}gt^2 - h\right)^2 = u^2t^2$$

$$\Rightarrow \quad \frac{1}{4}g^2t^4 - (gh + u^2)t^2 + (R^2 + h^2) = 0$$

For maximum range $t^2$ should be unique,

$$\therefore \quad R_{max} = \frac{u\sqrt{u^2 + 2gh}}{g}$$

**6.** Limiting friction,

$$f_l = \mu mg = 1.5 \text{ mN} \quad \text{...(i)}$$

Friction is the only force to provide centripetel force,

For first coin

$$F_{c_1} = m\omega^2 r_1$$

$$\omega = \frac{100}{3} \text{ rpm} = \frac{100}{3} \times \frac{2\pi}{60}$$

$$= 3.5 \text{ rad/s}$$

$$F_{c_1} = m \times (3.5)^2 \times 0.04 = 0.49 \text{ mN}.$$

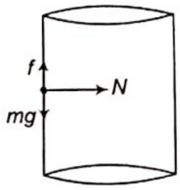
For second coin

$$F_{c_2} = m \times (3.5)^2 \times 0.14 = 1.72 \text{ mN}.$$

Here $\quad F_{c_1} < f_l < F_{c_2}$

Hence, first coin will rotate with the disc.

**7.** FBD of man is shown in figure




$$f = mg$$

and
$$N = m\omega^2 r$$

Also
$$f = \mu N$$

$$\therefore \quad mg = \mu m\omega^2 r$$

$$\omega = \sqrt{\frac{g}{\mu r}} = \sqrt{\frac{9.8}{0.15 \times 3}} = 4.65 \text{ rad/s}$$

**8.** Let $v_1$ be the combined speed of system, then

$$mv = (m + M)v_1$$

$$v_1 = \frac{mv}{m + M}$$

By conservation of energy

$$\frac{1}{2}mv^2 = \frac{1}{2}(m + M)v_1^2 + mgh$$

$$h = \frac{1}{2g}\left[v^2 - \frac{m}{m + M}v^2\right] = \frac{Mv^2}{2(m + M)g}$$

**9.** By conservation of energy

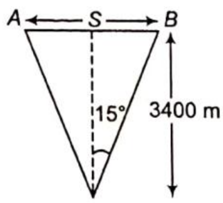
$$mg(h + x) = \frac{1}{2}kx^2$$

$$\frac{1}{2}kx^2 - mgx - mgh = 0$$

$$x = \frac{mg + \sqrt{(mg)^2 + 2kmgh}}{k}$$

$$= \frac{mg}{k}\left(1 + \sqrt{1 + \frac{2kh}{mg}}\right)$$

**10.** $s = 3400\sin 15°$




$$v = \frac{s}{t} = \frac{3400\sin 15°}{10}$$

$$= 340 \sin 15°$$

$$= 87.76 \text{ ms}^{-1}$$

**11.** Centripetal acceleration

$$a_c = \frac{v^2}{r} = \frac{(7.5)^2}{80} = 0.70$$

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$$a_T = 0.5 \text{ ms}^{-2}$$

$\therefore$ Net acceleration

$$a = \sqrt{a_c^2 + a_T^2}$$
$$= \sqrt{(0.7)^2 + (0.5)^2} = 0.86 \text{ ms}^{-2}$$

**12.** By using

$$S = ut + \frac{1}{2}at^2$$
$$6 = 2u + 2a$$
$$\Rightarrow \quad u + a = 3 \quad \text{...(i)}$$

and $\quad 6 + 24 = u(2 + u) + \frac{1}{2}a(2 + u)^2$

$$\Rightarrow \quad 30 = 6u + 18a$$
$$\Rightarrow \quad u + 3a = 5 \quad \text{...(ii)}$$

On solving,

$$a = 1 \text{ ms}^{-2}, u = 2 \text{ ms}^{-1}$$

$\therefore$ At the end of 8th s,

$$v = u + a(8) = 2 + 8 = 10 \text{ ms}^{-1}$$

**13.** We have

Power $\quad P = Fv \Rightarrow F = \frac{P}{v}$

Net force acting on the train

$$ma = F - kv = \frac{P}{v} - kv$$
$$\int_{v_0}^{2v_0} \frac{mv}{P - kv^2} dv = \int_0^t dt$$
$$\Rightarrow \quad -\frac{m}{k}[\ln|P - kv^2|]_{v_0}^{2v_0} = [t]_0^t$$
$$\Rightarrow \quad t = \frac{m}{k} \ln\left|\frac{P - kv_0^2}{P - 4kv_0^2}\right|$$

**14.** Same as question 7

$$\omega = \sqrt{\frac{g}{\mu r}}$$
$$= \sqrt{\frac{9.8}{0.3 \times 5}} = 2.55 \text{ rad/s}$$
$$= 2.55 \times \frac{60}{2\pi} = 24.36 \text{ rpm}$$

**15.** $a = p - qx$

$$\frac{dv}{dt} = p - qx$$
$$\Rightarrow \quad v\frac{dv}{dx} = p - qx$$
$$\frac{v^2}{2} = px = \frac{1}{2}qx^2$$
$$v^2 = 2px - qx^2$$

For distance $AB$, $v = 0$

$$0 = 2px - qx^2$$
$$\Rightarrow \quad x = 0, x = \frac{2p}{q}$$

For maximum velocity, $a = 0$

$$x = \frac{p}{q}$$
$$v^2 = \frac{2p^2}{q} - \frac{p^2}{q} = \frac{p^2}{q}$$
$$\Rightarrow \quad v = \frac{p}{\sqrt{q}}$$

**16.** By conservation of energy

Loss of PE = Gain in KE

$$mg\left(\frac{R}{2}\right) = \frac{1}{2}mv^2 - \frac{1}{2}v_0^2$$

By conservation of angular about given axis,

$$mv_0R = (mv\cos\theta)r$$

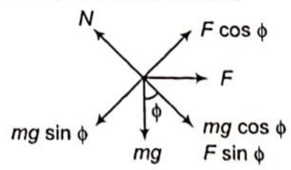
$r$ = radius of circular path at a distance $R/2$.

$$\therefore \quad r = \frac{\sqrt{3}}{2}$$
$$\Rightarrow \quad v\cos\theta = \frac{2v_0}{\sqrt{3}}$$

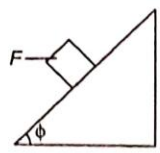
Vertical component,

$$v\sin\theta = \sqrt{\frac{gR - v_0^2}{3}}$$

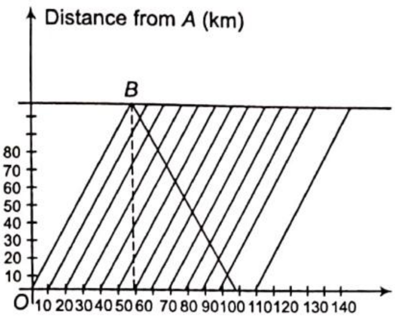
**17.** FBD of body is shown below,



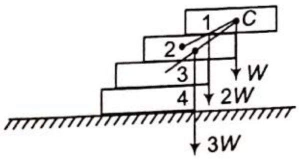

$$\therefore \quad N = mg\cos\phi + F\sin\phi$$




**18.** The number of cars the man meet in his way is equal to number of point of intersections *i.e.*, 13.




**19.** Refer to the figure, weight of each brick lies half way along its length since the bricks are homogeneous. The first top brick will still be in equilibrium wrt the second one when its centre of gravity is above the edge of the second brick, *i.e.*, $\frac{l}{2}$. The centre of gravity of the first and second bricks taken together will be at a distance $\frac{l}{4}$ from the edge of the second brick. This itself is the length by which the second brick may be overhanging wrt the third. The centre of gravity of three bricks lies on the line $AC$ and its position can be determined by using law of moments.

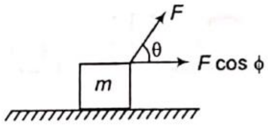



$$W\left(\frac{l}{2} - x\right) = 2Wx$$

$$\Rightarrow \qquad x = \frac{l}{2}$$

$\therefore$ The lengths of the overhanging edges of the bricks will be $\frac{l}{2}, \frac{l}{4}$ and $\frac{l}{6}$.

**20.** By Newton's second law of motion,




$$F\cos\theta = ma$$

$$\Rightarrow \qquad 2mb\cos(a + bs) = m\frac{dv}{dt}$$

$$\Rightarrow \qquad \frac{dv}{dt} = 2b\cos(a + bs)$$

$$\Rightarrow \qquad v\frac{dv}{ds} = 2b\cos(a + bs)$$

$$\Rightarrow \qquad \int_0^v v\,dv = 2b\int_0^s \cos(a + bs)\,ds$$

$$\Rightarrow \qquad \left[\frac{v^2}{2}\right]_0^v = \left[\frac{2b\sin(a + bs)}{b}\right]_0^s$$

$$\Rightarrow \qquad v^2 = u[\sin(a + bs) - \sin a]$$

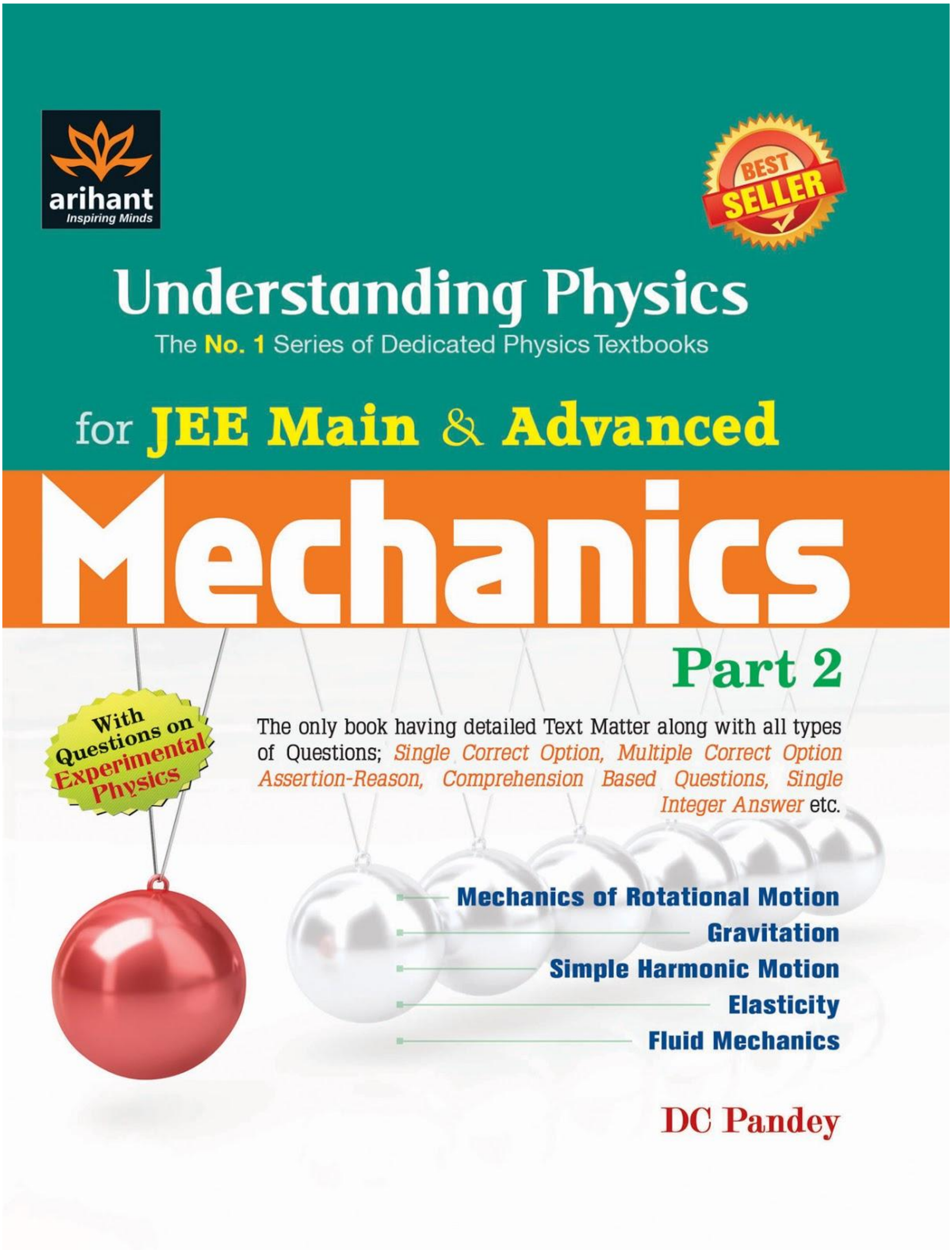
$$\Rightarrow \qquad v = 2(\sin\theta - \sin a)^{1/2} \quad (\because \theta = a + bs)$$

# For More Practice

# Chapter 9 Mechanics of Rotational Motion

# From " Understanding Physics " Mechanics Part 2

# By D C Pandey Arihant is included here

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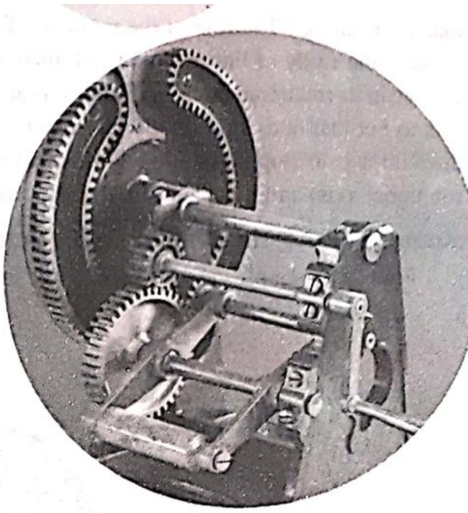


# 9

# Mechanics of Rotational Motion

## Chapter Contents

## 9.1 Moment of Inertia

Like the centre of mass, the moment of inertia is a property of an object that is related to its mass distribution. The moment of inertia (denoted by $I$) is an important quantity in the study of system of particles that are rotating . The role of the moment of inertia in the study of rotational motion is analogous to that of mass in the study of linear motion. Moment of inertia gives a measurement of the resistance of a body to a change in its rotational motion. If a body is at rest, the larger the moment of inertia of a body, the more difficult it is to put that body into rotational motion. Similarly, the larger the moment of inertia of a body, the more difficult it is to stop its rotational motion. The moment of inertia is calculated about some axis (usually the rotational axis) and it depends on the mass as well as its distribution about that axis.

### Moment of Inertia of a Single Particle

For a very simple case the moment of inertia of a single particle about an axis is given by,

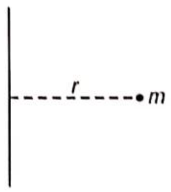



Fig. 9.1

$$I = mr^2 \qquad \ldots\text{(i)}$$

Here, $m$ is the mass of the particle and $r$ its distance from the axis under consideration.

### Moment of Inertia of a System of Particles

The moment of inertia of a system of particles about an axis is given by,

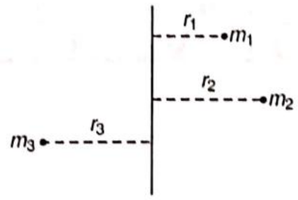



Fig. 9.2

$$I = \sum_i m_i r_i^2 \qquad \ldots\text{(ii)}$$

where $r_i$ is the perpendicular distance from the axis to the $i$th particle, which has a mass $m_i$.

### Moment of Inertia of Rigid Bodies

For a continuous mass distribution such as found in a rigid body, we replace the summation of Eq. (ii) by an integral. If the system is divided into infinitesimal elements of mass $dm$ and if $r$ is the distance from a mass element to the axis of rotation, the moment of inertia is,

$$I = \int r^2 \, dm$$

where the integral is taken over the system.

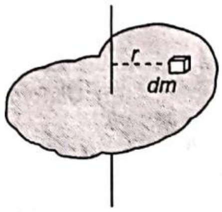



**Fig. 9.3**

## Moment of Inertia of a Uniform Cylinder

Let us find the moment of inertia of a uniform cylinder about an axis through its centre of mass and perpendicular to its base. Mass of the cylinder is $M$ and radius is $R$.

We first divide the cylinder into annular shells of width $dr$ and length $l$ as shown in figure. The moment of inertia of one of these shells is

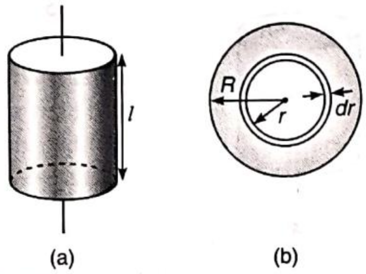



**Fig. 9.4**

$$dI = r^2 dm = r^2\ (\rho \cdot dV)$$

Here, $\rho$ = density of cylinder

and $dV$ = volume of shell $= 2\pi rl\ dr$

$\therefore \quad dI = 2\pi\rho l\, r^3\ dr$

The cylinder's moment of inertia is found by integrating this expression between 0 and $R$,

So, $$I = 2\pi\rho l \int_0^R r^3\, dr = \frac{\pi\rho l}{2} R^4 \qquad \text{...(iii)}$$

The density $\rho$ of the cylinder is the mass divided by the volume.

$\therefore$ $$\rho = \frac{M}{\pi R^2 l} \qquad \text{...(iv)}$$

From Eqs. (iii) and (iv), we have

$$I = \frac{1}{2} MR^2$$

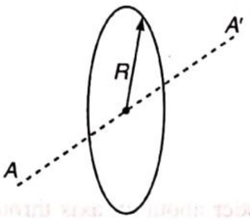
Proceeding in the similar manner we can find the moment of inertia of certain rigid bodies about some given axis. Moments of inertia of several rigid bodies with symmetry are listed in Table. 9.1.

In all cases except ($f$) the rotational axis $AA'$ passes through the centre of mass.

Table 9.1

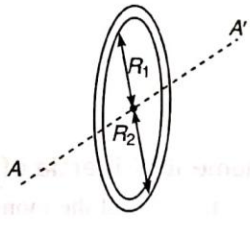
(a) Very thin circular hoop

$$I = MR^2$$



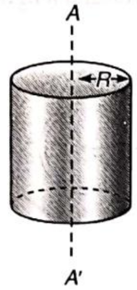

(b) Uniform circular hoop

$$I = M\,\frac{R_1^2 + R_2^2}{2}$$



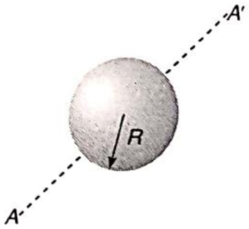

(c) Uniform solid cylinder

$$I = \frac{1}{2}MR^2$$



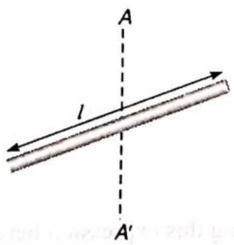

(d) Uniform solid sphere

$$I = \frac{2}{5}MR^2$$



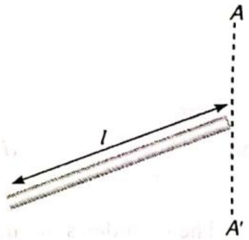

(e) Uniform thin rod

$$I = \frac{1}{12}Ml^2$$



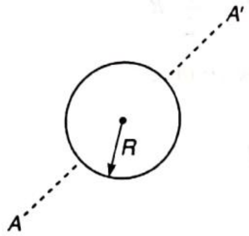

(f) Uniform thin rod

$$I = \frac{1}{3}Ml^2$$



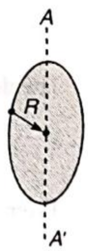

(g) Very thin spherical shell

$$I = \frac{2}{3}MR^2$$



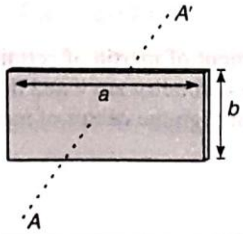

(h) Thin circular sheet

$$I = \frac{1}{4}MR^2$$



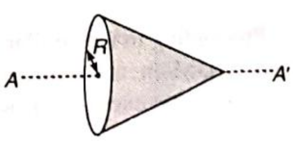

(i) Thin rectangular sheet

$$I = M\,\frac{a^2 + b^2}{12}$$




(j) Uniform right cone

$$I = \frac{3}{10}MR^2$$



## Theorems on Moment of Inertia

There are two important theorems on moment of inertia, which, in some cases, enable the moment of inertia of a body to be determined about an axis, if its moment of inertia about some other axis is known. Let us now discuss both of them.

### (i) Theorem of parallel axes

A very useful theorem, called the parallel axes theorem relates the moment of inertia of a rigid body about two parallel axes, one of which passes through the centre of mass.

Two such axes are shown in figure for a body of mass $M$. If $r$ is the distance between the axes and $I_{COM}$ and $I$ are the respective moments of inertia about them, these moments are related by,

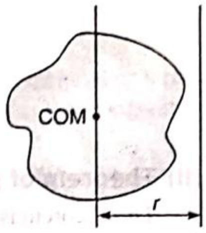



**Fig. 9.5**

$$I = I_{COM} + Mr^2$$

We now present a proof of the above theorem.

**Proof :** A cross-section view of a rigid body is shown in Fig. 9.6. The body is oriented so that the two axes, one through COM (the centre of mass) and the other through $A$, are perpendicular to the plane of the pages. The coordinates of these two points are $(x_{COM}, y_{COM}, 0)$ and $(x_{COM} + a, y_{COM} + b, 0)$ respectively. The distance between the two axes is

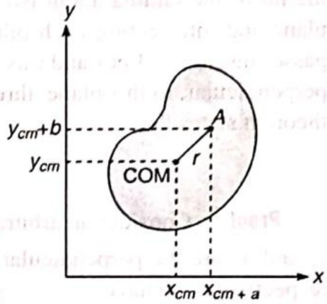



**Fig. 9.6**

$$r^2 = a^2 + b^2$$

The moment of inertia around the axis through $CM$ is

$$I_{COM} = \sum_i m_i [(x_i - x_{COM})^2 + (y_i - y_{COM})^2]$$

and the moment of inertia around the parallel axis through $A$ is,

$$I_A = \sum_i m_i [(x_i - x_{COM} - a)^2 + (y_i - y_{COM} - b)^2]$$

Which after some rearrangement, can be written as

$$I_A = \sum_i m_i [(x_i - x_{COM})^2 + (y_i - y_{COM})^2]$$

$$- 2a \sum_i m_i (x_i - x_{COM}) - 2b \sum_i m_i (y_i - y_{COM}) + (a^2 + b^2) \sum_i m_i$$

The first of these terms is $I_{COM}$. The second and third terms are zero from the definition of the centre of mass.

$$\sum_i m_i x_i = x_{COM} \sum_i m_i$$

and

$$\sum_i m_i y_i = y_{COM} \sum_i m_i$$

Since, $r^2 = a^2 + b^2$ and $M = \sum_i m_i$,

The fourth term is $Mr^2$. Thus,

$$I_A = I_{cm} + 0 + 0 + Mr^2 \qquad \text{or} \qquad I_A = I_{cm} + Mr^2$$

which is the parallel axis theorem.

**Note** *From the above theorem we can see that among too many parallel axes moment of inertia is least about an axis which passes through centre of mass. e.g.,* $I_2$ *is least among* $I_1$, $I_2$ *and* $I_3$. *Similarly,* $I_5$ *is least among* $I_4$, $I_5$ *and* $I_6$.

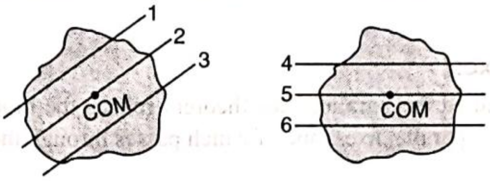



Fig. 9.7

## (ii) Theorem of perpendicular axes

This theorem is applicable only to the plane bodies (two dimensional). The theorem states that the moment of inertia of a plane lamina about an axis perpendicular to the plane of the lamina is equal to the sum of the moments of inertia of the lamina about two axes perpendicular to each other, in its own plane and intersecting each other, at the point where the perpendicular axis passes through it. Let $x$ and $y$ axes be chosen in the plane of the body and $z$-axis perpendicular, to this plane, three axes being mutually perpendicular, then the theorem states that

$$I_z = I_x + I_y$$

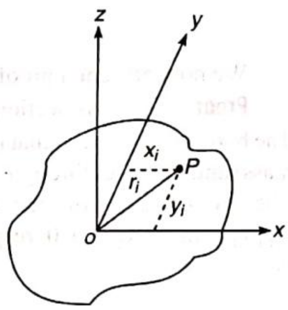



Fig. 9.8

**Proof :** Consider an arbitrary particle $P$ of mass $m_i$, distant $r_i$ from $o$ and $x_i$ and $y_i$ are the perpendicular distances of point $P$ from the axes $oy$ and $ox$ respectively, we have

$$I_z = \sum_i m_i r_i^2, \quad I_x = \sum_i m_i y_i^2 \quad \text{and} \quad I_y = \sum_i m_i x_i^2$$

So that,

$$I_x + I_y = \sum_i m_i y_i^2 + \sum_i m_i x_i^2$$

$$= \sum_i m_i (y_i^2 + x_i^2)$$

$$= \sum_i m_i r_i^2$$

$$= I_z$$

*i.e.,*

$$I_z = I_x + I_y$$

### Radius of Gyration

Radius of gyration ($K$) of a body about an axis is the effective distance from this axis where the whole mass can be assumed to be concentrated so that the moment of inertia remains the same. Thus,

$$I = MK^2$$

or

$$K = \sqrt{\frac{I}{M}}$$

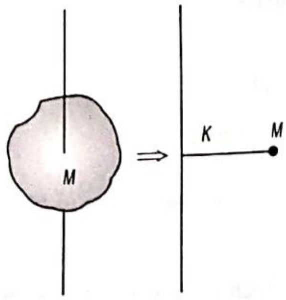



Fig. 9.9

*e.g.*, radius of gyration of a disc about an axis perpendicular to its plane and passing through its centre of mass is

$$K = \sqrt{\frac{\frac{1}{2}MR^2}{M}} = \frac{R}{\sqrt{2}}$$

## Important points in Moment Of Inertia

- Theorem of parallel axes is applicable for any type of rigid body whether it is a two dimensional or three dimensional, while the theorem of perpendicular axes is applicable for laminar type or two dimensional bodies only.
- In theorem of perpendicular axes, the point of intersection of the three axes ($x$, $y$ and $z$) may be any point on the plane of body (it may even lie outside the body). This point may or may not be the centre of mass of the body.
- Moment of inertia of a part of a rigid body (symmetrically cut from the whole mass) is the same as that of the whole body. *e.g.*, in figure (a) moment of inertia of the section shown (a part of a circular disc) about an axis perpendicular to its plane and passing through point $O$ is $\frac{1}{2}MR^2$ as the moment of inertia of the complete disc is also $\frac{1}{2}MR^2$. This can be shown as in Fig. 9.10.

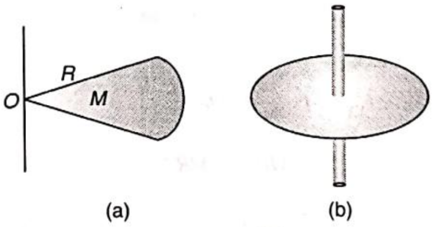



Fig. 9.10

Suppose the given section is $\frac{1}{n}$ th part of the disc, then mass of the disc will be $nM$.

$$I_{\text{disc}} = \frac{1}{2}(nM)R^2$$

$$\therefore \quad I_{\text{section}} = \frac{1}{n} I_{\text{disc}} = \frac{1}{2} MR^2$$

**Sample Example 9.1** *Three rods each of mass m and length l are joined together to form an equilateral triangle as shown in figure. Find the moment of inertia of the system about an axis passing through its centre of mass and perpendicular to the plane of the triangle.*

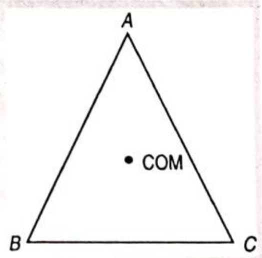



Fig. 9.11

**Solution** Moment of inertia of rod $BC$ about an axis perpendicular to plane of triangle $ABC$ and passing through the mid-point of rod $BC$ (*i.e.*, $D$) is

$$I_1 = \frac{ml^2}{12}$$

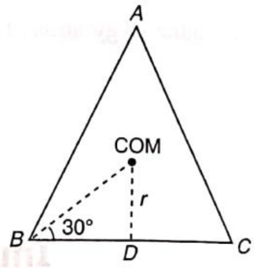


Fig. 9.12

From theorem of parallel axes, moment of inertia of this rod about the asked axis is

$$I_2 = I_1 + mr^2 = \frac{ml^2}{12} + m\left(\frac{l}{2\sqrt{3}}\right)^2 = \frac{ml^2}{6}$$

$\therefore$ Moment of inertia of all the three rods is

$$I = 3I_2 = 3\left(\frac{ml^2}{6}\right) = \frac{ml^2}{2}$$

**Sample Example 9.2** *Find the moment of inertia of a solid sphere of mass M and radius R about an axis XX shown in figure.*

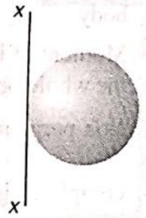


Fig. 9.13

**Solution** From theorem of parallel axis,

$$I_{XX} = I_{COM} + Mr^2$$

$$= \frac{2}{5}MR^2 + MR^2$$

$$= \frac{7}{5}MR^2$$

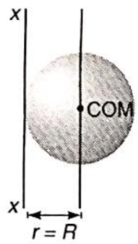


Fig. 9.14

**Sample Example 9.3** *Consider a uniform rod of mass m and length 2l with two particles of mass m each at its ends. Let AB be a line perpendicular to the length of the rod and passing through its centre. Find the moment of inertia of the system about AB.*

**Solution**

$$I_{AB} = I_{\text{rod}} + I_{\text{both particles}}$$

$$= \frac{m(2l)^2}{12} + 2(ml^2)$$

$$= \frac{7}{3}ml^2$$

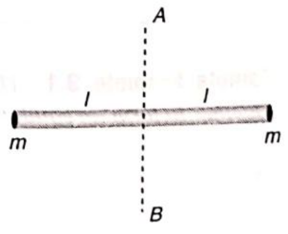


Fig. 9.15

**Sample Example 9.4** *Find the moment of inertia of the rod AB about an axis YY as shown in figure. Mass of the rod is m and length is l.*

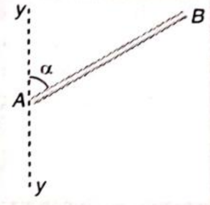



Fig. 9.16

**Solution** Mass per unit length of the rod $=\dfrac{m}{l}$

Mass of an element $PQ$ of the rod is, $dm=\left(\dfrac{m}{l}\right)dx$

Perpendicular distance of this elemental mass about $yy$ is $r=x\sin\alpha$

$\therefore$ Moment of inertia of this small element of the rod (can be assumed as a point mass) about $yy$ is,

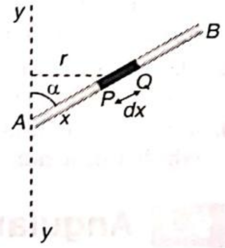



Fig. 9.17

$$dI=(dm)r^2=\left(\frac{m}{l}dx\right)(x\sin\alpha)^2$$

$$=\frac{m}{l}\sin^2\alpha\, x^2\, dx$$

$\therefore$ Moment of inertia of the complete rod,

$$I=\int_{x=0}^{x=l} dI=\frac{m}{l}\sin^2\alpha\int_0^l x^2\,dx=\frac{ml^2}{3}\sin^2\alpha \qquad \textbf{Ans.}$$

**Note** *(i)* $I=0$ *if* $\alpha=0$ *(ii)* $I=\dfrac{ml^2}{3}$ *if* $\alpha=\dfrac{\pi}{2}$ *or* $90°$

## Introductory Exercise 9.1

**1.** About what axis would a uniform cube have its minimum moment of inertia?

**2.** If $I_1$ is the moment of inertia of a thin rod about an axis perpendicular to its length and passing through its centre of mass and $I_2$ the moment of inertia of the ring formed by the same rod about an axis passing through the centre of mass of the ring and perpendicular to the plane of the ring. Then find the ratio $\dfrac{I_1}{I_2}$.

**3.** Find the radius of gyration of a rod of mass $m$ and length $2l$ about an axis passing through one of its ends and perpendicular to its length.

**4.** There are four solid balls with their centres at the four corners of a square of side $a$. The mass of each sphere is $m$ and radius is $r$. Find the moment of inertia of the system about (i) one of the sides of the square (ii) one of the diagonals of the square.

**5.** A non-uniform rod $AB$ has a mass $M$ and length $2l$. The mass per unit length of the rod is $mx$ at a point of the rod distant $x$ from $A$. Find the moment of inertia of this rod about an axis perpendicular to the rod (a) through $A$ (b) through the mid-point of $AB$.

**6.** A circular lamina of radius $a$ and centre $O$ has a mass per unit area of $kx^2$, where $x$ is the distance from $O$ and $k$ is a constant. If the mass of the lamina is $M$, find in terms of $M$ and $a$, the moment of inertia of the lamina about an axis through $O$ and perpendicular to the lamina.

**7.** The uniform disc shown in the figure has a moment of inertia of 0.6 kg-m$^2$ around the axis that passes through $O$ and is perpendicular to the plane of the page. If a segment is cut out from the disc as shown, what is the moment of inertia of the remaining disc?


**Fig. 9.18**

**8.** Particles of masses $1\text{ g}, 2\text{ g}, 3\text{ g}, \ldots, 100\text{ g}$ are kept at the marks $1\text{ cm}, 2\text{ cm}, 3\text{ cm}, \ldots, 100\text{ cm}$ respectively on a metre scale. Find the moment of inertia of the system of particles about a perpendicular bisector of the metre scale.

**9.** The radius of gyration of a uniform disc about a line perpendicular to the disc equals its radius $R$. Find the distance of the line from the centre.

**10.** If two circular disks of the same weight and thickness are made from metals having different densities. Which disk, if either will have the larger moment of inertia about its central axis.

## 9.2 Angular Velocity

The term angular velocity $(\vec{\omega})$ is defined for a particle about a point.

Suppose a particle $P$ is moving with a velocity $\vec{\mathbf{v}}$, its position vector at some moment of time $t = t$, is $\vec{\mathbf{r}}$ with respect to a fixed point $O$. At time $t = t + dt$ the radius vector becomes $\vec{\mathbf{r}} + d\vec{\mathbf{r}}$. It has been rotated an angle $d\theta$ in time $dt$. Then the angular speed of particle $P$ about point $O$, *i.e.*,

$$\omega = \frac{d\theta}{dt}$$

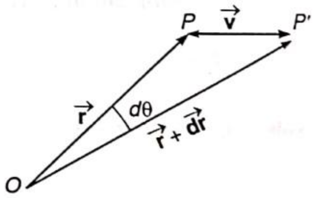



**Fig. 9.19**

This is also equal to the component of velocity perpendicular to $\vec{\mathbf{r}}$ divided by the distance of particle $P$ from point $O$ at that instant or,

$$\omega = \frac{v_\perp}{r}$$

In vector form the linear velocity, the angular velocity and the radius vector are related by,

$$\vec{\mathbf{v}} = \vec{\omega} \times \vec{\mathbf{r}}$$

Thus, angular velocity may be represented as a vector quantity whose direction is perpendicular to the plane of motion given by the right hand rule.

## Important Points in Angular Velocity

- If a particle is moving in a circle it is in pure rotational motion about the centre of the circle, while for a moment it may be in pure translational motion about some other point.

  If a particle $P$ is moving in a circle, its angular velocity about centre of the circle ($\omega_C$ ) is two times the angular velocity about any point on the circumference of the circle ($\omega_O$ )

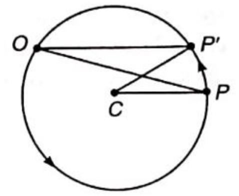



Fig. 9.20

or
$$\omega_C = 2\omega_O$$

This is because $\angle P'CP = 2\angle P'OP$ (by property of a circle)

$$\omega_C = \frac{\angle P'CP}{t_{pp'}}, \quad \omega_O = \frac{\angle P'OP}{t_{pp'}}$$

From these relations we can see that $\omega_C = 2\omega_O$.

- If a rigid body is rotating about a fixed axis with angular speed $\omega$, all the particles in rigid body rotate same angle in same interval of time, *i.e.*, their angular speed is same ($\omega$). They rotate in different circles of different radii. The planes of these circles are perpendicular to the rotational axis. Linear speeds of different particles are different. Linear speed of a particle situated at a distance $r$ from the rotational axis is

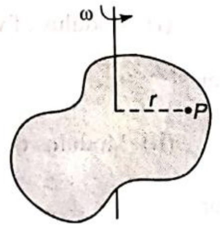



Fig. 9.21

$$v = r\omega$$

or
$$v \propto r$$

- Angular velocity of a rigid body ($\omega$) is $\frac{d\theta}{dt}$. Here $\theta$ is the angle between the line joining any two points (say $A$ and $B$) on the rigid body and any reference line (dotted) as shown in figure.

  For example $AB$ is a rod of length 4 m. End $A$ is resting against a vertical wall $OY$ and $B$ is moving towards right with constant speed $v_B = 10\,\text{m/s}$. To find the angular speed of rod at $\theta = 30°$, we can proceed as under.

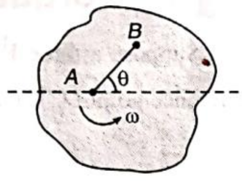



Fig. 9.22

$$OB = x = AB\cos\theta$$

$\therefore$
$$x = 4\cos\theta$$

or
$$\frac{dx}{dt} = -4\sin\theta\left(\frac{d\theta}{dt}\right)$$

$\therefore$
$$\left(\frac{d\theta}{dt}\right) = -\frac{(dx/dt)}{4\sin\theta} \quad \left(\frac{dx}{dt} = v_B = 10\,\text{m/s}\right)$$

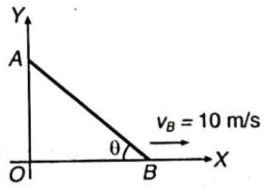



Fig. 9.23

or
$$\omega = -\frac{10}{4\sin 30°} = -5\,\text{rad/s}$$

Here, negative sign implies that $\theta$ decreases as $t$ increases $\left(\frac{d\theta}{dt} < 0\right)$.

**Sample Example 9.5** *A particle A moves along a circle of radius $R = 10$ cm so that its radius vector $\vec{\mathbf{r}}$ relative to O rotates with constant angular velocity $\omega = 0.2$ rad/s. Find the modulus of the velocity of the particle and modulus and direction of its total acceleration.*

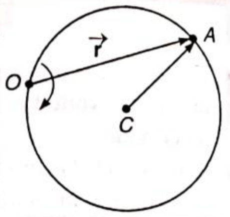



Fig. 9.24

**Solution** Given that $\omega_O = 0.2$ rad/s, $R = 10\,\text{cm} = 0.1\,\text{m}$

$\therefore$ $$\omega_C = 2\omega_O = 0.4 \text{ rad/s}$$

(i) Modulus of velocity $|\vec{\mathbf{v}}| = R\omega_C = (0.1)(0.4)$

or $$|\vec{\mathbf{v}}| = 0.04 \text{ m/s} \quad \text{or} \quad 4 \text{ cm/s}$$

(ii) Modulus of total acceleration $|\vec{\mathbf{a}}| = R\omega_C^2$

or $$|\vec{\mathbf{a}}| = (0.1)(0.4)^2 = 0.016 \text{ m/s}^2$$

or $$|\vec{\mathbf{a}}| = 1.6 \text{ cm/s}^2$$

(iii) The direction of its total acceleration (centripetal acceleration) will be towards centre $C$.

## 9.3 Torque

Suppose a force $\vec{\mathbf{F}}$ is acting on a particle $P$ and let $\vec{\mathbf{r}}$ be the position vector of this particle about some reference point $O$. The torque of this force $\vec{\mathbf{F}}$, about $O$ is defined as,

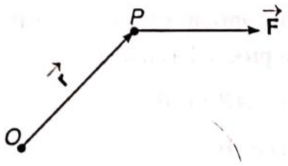



Fig. 9.25

$$\vec{\tau} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}$$

This is a vector quantity having its direction perpendicular to both $\vec{\mathbf{r}}$ and $\vec{\mathbf{F}}$ according to the rule of cross product.

**Note** *Here,* $\vec{\mathbf{r}} = \vec{\mathbf{r}}_F - \vec{\mathbf{r}}_O$

$\vec{\mathbf{r}}_F$ = *position vector of point, where force is acting and*

$\vec{\mathbf{r}}_O$ = *position vector of point about which torque is required. See Sample Example* 9.8.

### Torque of a force about a line

Consider a rigid body rotating about a fixed axis $AB$. Let $\vec{F}$ be a force acting on the body at point $P$. Take the origin $O$ somewhere on the axis of rotation. The torque of $\vec{F}$ about $O$ is

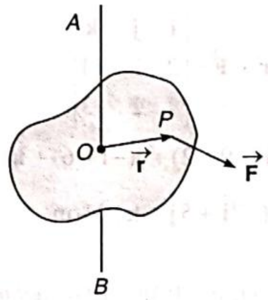



**Fig. 9.26**

$$\vec{\tau} = \vec{r} \times \vec{F}$$

Its component along $AB$ is called the torque of $\vec{F}$ about $AB$.

## Important Points in Torque

- When a rigid body is rotating about a fixed axis and a force is applied on it at some point then we are concerned with the component of torque of this force about the axis of rotation not with the net torque.
- The component of torque about axis of rotation is independent of the choice of the origin $O$, so long as it is chosen on the axis of rotation, *i.e.,* we may choose point $O$ anywhere on the line $AB$.
- Component of torque along axis of rotation $AB$ is zero if

  (a) $\vec{F} \parallel AB$

  (b) $\vec{F}$ intersects $AB$ at some point
- If $\vec{F}$ is perpendicular to $AB$, but does not intersect it, then component of torque about line $AB$ = magnitude of force $\vec{F}$ × perpendicular distance of $\vec{F}$ from the line $AB$ (called the lever arm or moment arm) of this torque.
- If there are more than one force $\vec{F}_1, \vec{F}_2, \ldots$ acting on a body, the total torque will be

$$\vec{\tau} = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \ldots$$

  But if the forces act on the same particle, one can add the forces and then take the torque of the resultant force, or

$$\vec{\tau} = \vec{r} \times (\vec{F}_1 + \vec{F}_2 + \ldots)$$

**Sample Example 9.6** *Find the torque of a force* $\vec{\mathbf{F}} = (\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}})$ *N about a point O. The position vector of point of application of force about O is* $\vec{\mathbf{r}} = (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}})$ *m.*

**Solution** Torque

$$\vec{\tau} = \vec{\mathbf{r}} \times \vec{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 3 & -1 \\ 1 & 2 & -3 \end{vmatrix}$$

$$= \hat{\mathbf{i}}(-9+2) + \hat{\mathbf{j}}(-1+6) + \hat{\mathbf{k}}(4-3)$$

or

$$\vec{\tau} = (-7\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + \hat{\mathbf{k}}) \text{ N-m}$$

**Sample Example 9.7** *A small ball of mass 1.0 kg is attached to one end of a 1.0 m long massless string and the other end of the string is hung from a point. When the resulting pendulum is 30° from the vertical, what is the magnitude of torque about the point of suspension. [Take g = 10 m/s²]*

**Solution** Two forces are acting on the ball :

(i) tension ($T$)

(ii) weight ($mg$)

Torque of tension about point $O$ is zero, as it passes through $O$.

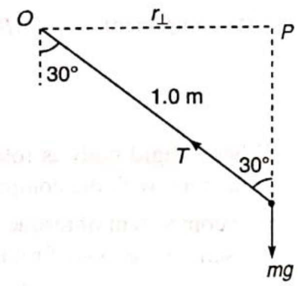



Fig. 9.27

$$\tau_{mg} = F \times r_\perp$$

Here, $r_\perp = OP = 1.0 \sin 30° = 0.5$ m

$\therefore$

$$\tau_{mg} = (mg)(0.5)$$

$$= (1)(10)(0.5)$$

$$= 5 \text{ N-m}$$

**Sample Example 9.8** *A force* $\vec{\mathbf{F}} = (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 4\hat{\mathbf{k}})$ *N is acting at point (2 m, −3 m, 6 m). Find torque of this force about a point whose position vector is* $(2\hat{\mathbf{i}} - 5\hat{\mathbf{j}} + 3\hat{\mathbf{k}})$ *m.*

**Solution**

$$\vec{\tau} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}$$

Here,

$$\vec{\mathbf{r}} = \vec{\mathbf{r}}_F - \vec{\mathbf{r}}_0$$

$$= (2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}) - (2\hat{\mathbf{i}} - 5\hat{\mathbf{j}} + 3\hat{\mathbf{k}})$$

$$= (2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) \text{ m}$$

Now,

$$\vec{\tau} = \vec{\mathbf{r}} \times \vec{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 2 & 3 \\ 2 & 3 & -4 \end{vmatrix} = (-17\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - 4\hat{\mathbf{k}}) \text{ N-m}$$

## 9.4 Rotation of a Rigid Body about a Fixed Axis

When a body is rotating about a fixed axis, any point $P$ located in the body travels along a circular path. Before, analysing the circular motion of point $P$, we will first study the angular motion properties of a rigid body.

### Angular motion

Since, a point is without dimension, it has no angular motion. Only lines or bodies undergo angular motion. Let us consider the angular motion of a radial line $r$ located with the shaded plane.

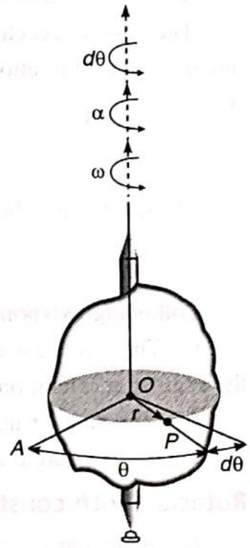


Fig. 9.28

### Angular position

The angular position of $r$ is defined by the angle $\theta$, measured between a fixed reference line $OA$ and $r$.

### Angular displacement

The change in the angular position, often measured as a differential $d\vec{\theta}$ is called the angular displacement. (Finite angular displacements are not vector quantities, although differential rotations $d\vec{\theta}$ are vectors). This vector has a magnitude $d\theta$ and the direction of $d\vec{\theta}$ is along the axis.

Specifically, the direction of $d\vec{\theta}$ is determined by right hand rule; that is, the fingers of the right hand are curled with the sense of rotation, so that in this case the thumb or $d\vec{\theta}$ points upward.

### Angular velocity

The time rate of change in the angular position is called the angular velocity $\vec{\omega}$. Thus,

$$\omega = \frac{d\theta}{dt} \qquad \text{...(i)}$$

It is expressed here in scalar form, since its direction is always along the axis of rotation, *i.e.*, in the same direction as $d\vec{\theta}$.

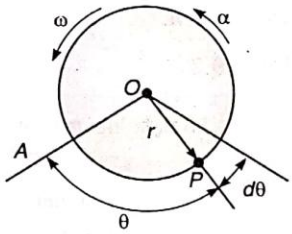


Fig. 9.29

### Angular acceleration

The angular acceleration $\vec{\alpha}$ measures the time rate of change of the angular velocity. Hence, the magnitude of this vector may be written as,

$$\alpha = \frac{d\omega}{dt} \qquad \text{...(ii)}$$

It is also possible to express $\alpha$ as,

$$\alpha = \frac{d^2\theta}{dt^2}$$

The line of action of $\vec{\alpha}$ is the same as that for $\vec{\omega}$, however its sense of direction depends on whether $\omega$ is increasing or decreasing with time. In particular, if $\omega$ is decreasing, $\vec{\alpha}$ is called an angular deceleration and therefore, has a sense of direction which is opposite to $\vec{\omega}$.

### Torque and angular acceleration for a rigid body

The angular acceleration of a rigid body is directly proportional to the sum of the torque components along the axis of rotation. The proportionality constant is the inverse of the moment of inertia about that axis, or

$$\alpha = \frac{\Sigma\tau}{I}$$

Thus, for a rigid body we have the rotational analog of Newton's second law :

$$\Sigma\tau = I\alpha \qquad \text{...(iii)}$$

Following two points are important regarding the above equation.

(i) The above equation is valid only for rigid bodies. If the body is not rigid like a rotating tank of water, the angular acceleration $\alpha$ is different for different particles.

(ii) The sum $\Sigma\tau$ in the above equation includes only the torques of the external forces, because all the internal torques add to zero.

### Rotation with constant angular acceleration

If the angular acceleration of the body is constant then Eqs. (i) and (ii) when integrated yield a set of formulae which relate the body's angular velocity, angular position and time. These equations are similar to equations used for rectilinear motion. Table given ahead compares the linear and angular motion with constant acceleration.

Table 9.2

| Straight line motion with constant linear acceleration | Fixed axis rotation with constant angular acceleration |
|---|---|
| $a$ = constant | $\alpha$ = constant |
| $v = u + at$ | $\omega = \omega_0 + \alpha t$ |
| $s = s_0 + ut + \frac{1}{2}at^2$ | $\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$ |
| $v^2 = u^2 + 2a(s - s_0)$ | $\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$ |

Here $\theta_0$ and $\omega_0$ are the initial values of the body's angular position and angular velocity respectively.

### Kinetic Energy of a rigid body rotating about a fixed axis

Suppose a rigid body is rotating about a fixed axis with angular speed $\omega$. Then, kinetic energy of the rigid body will be :

$$K = \sum_i \frac{1}{2} m_i v_i^2 = \sum_i \frac{1}{2} m_i (\omega r_i)^2$$

$$= \frac{1}{2}\omega^2 \sum_i m_i r_i^2$$

$$= \frac{1}{2} I\omega^2 \qquad \left(\text{as } \sum_i m_i r_i^2 = I\right)$$

Thus, $$\text{KE} = \frac{1}{2} I\omega^2$$

Sometimes it is called the rotational kinetic energy.

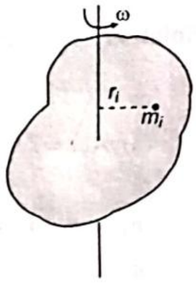



**Fig. 9.30**

**Sample Example 9.9** *A solid sphere of mass 2 kg and radius 1 m is free to rotate about an axis passing through its centre. Find a constant tangential force F required to rotate the sphere with 10 rad/s in 2 s. Also find the number of rotations made by the sphere in that time interval.*

**Solution** Since, the force is constant, the torque produced by it and the angular acceleration $\alpha$ will be constant. Hence, we can apply

$$\omega = \omega_0 + \alpha t \quad \text{etc.}$$

$$10 = 0 + (\alpha)(2)$$

$\therefore$ $$\alpha = 5\ \text{rad/s}^2$$

Further, the force is tangential. Therefore, the perpendicular distance from the axis of rotation will be equal to the radius of the sphere.

$\therefore$ $$\alpha = \frac{\tau}{I} = \frac{F \cdot R}{\frac{2}{5} mR^2} = \frac{5F}{2mR}$$

or $$F = \frac{2mR\alpha}{5}$$

Substituting the value, we have

$$F = \frac{(2)(2)(1)(5)}{(5)} = 4\ \text{N}$$

Further, Angle rotated $$\theta = \frac{1}{2}\alpha t^2 = \frac{1}{2}(5)(2)^2$$

$$= 10\ \text{rad}$$

$\therefore$ Number of rotations $$n = \frac{\theta}{2\pi} = \frac{10}{2\pi} = \frac{5}{\pi}$$

**Sample Example 9.10** *The angular position of a point on the rim of a rotating wheel is given by $\theta = 4t - 3t^2 + t^3$, where $\theta$ is in radians and $t$ is in seconds. What are the angular velocities at*

*(a) $t = 2.0\,s$ and*

*(b) $t = 4.0\,s$ ?*

*(c) What is the average angular acceleration for the time interval that begins at $t = 2.0$ s and ends at $t = 4.0$ s?*

*(d) What are the instantaneous angular acceleration at the beginning and the end of this time interval?*

**Solution** Angular velocity $\omega = \frac{d\theta}{dt} = \frac{d}{dt}(4t - 3t^2 + t^3)$

or $\omega = 4 - 6t + 3t^2$

(a) At $t = 2.0$ s, $\omega = 4 - 6 \times 2 + 3(2)^2$

or $\omega = 4$ rad/s

(b) At $t = 4.0$ s, $\omega = 4 - 6 \times 4 + 3(4)^2$

or $\omega = 28$ rad/s

(c) Average angular acceleration

$$\alpha_{av} = \frac{\omega_f - \omega_i}{t_f - t_i} = \frac{28 - 4}{4 - 2}$$

or $\alpha_{av} = 12$ rad/s$^2$

(d) Instantaneous angular acceleration is,

$$\alpha = \frac{d\omega}{dt} = \frac{d}{dt}(4 - 6t + 3t^2)$$

or $\alpha = -6 + 6t$

At $t = 2.0$ s, $\alpha = -6 + 6 \times 2 = 6$ rad/s$^2$

At $t = 4.0$ s, $\alpha = -6 + 6 \times 4 = 18$ rad/s$^2$

## Introductory Exercise 9.2

1. A body rotates about a fixed axis with an angular acceleration 1 rad/s$^2$. Through what angle does it rotates during the time in which its angular velocity increases from 5 rad/s to 15 rad/s.
2. A wheel starting from rest is uniformly accelerated at 4 rad/s$^2$ for 10 s. It is allowed to rotate uniformly for the next 10 s and is finally brought to rest in the next 10 s. Find the total angle rotated by the wheel.
3. A flywheel of moment of inertia 5.0 kg m$^2$ is rotated at a speed of 10 rad/s. Because of the friction at the axis it comes to rest in 10 s. Find the average torque of the friction.
4. A wheel of mass 10 kg and radius 0.2 m is rotating at an angular speed of 100 rpm, when the motion is turned off. Neglecting the friction at the axis, calculate the force that must be applied tangentially to the wheel to bring it to rest in 10 rev. Assume wheel to be a disc.
5. A solid body rotates about a stationary axis according to the law $\theta = 6t - 2t^3$. Here, $\theta$ is in radian and $t$ in seconds. Find :
   (a) the mean values of the angular velocity and angular acceleration averaged over the time interval between $t = 0$ and the complete stop,
   (b) the angular acceleration at the moment when the body stops.

**Hint** *If $y = y(t)$, then mean/average value of $y$ between $t_1$ and $t_2$ is* $\langle y \rangle = \frac{\int_{t_1}^{t_2} y(t)\,dt}{t_2 - t_1}$.

6. A solid body starts rotating about a stationary axis with an angular acceleration $\alpha = (2.0 \times 10^{-2})t$ rad/s$^2$, here, $t$ is in seconds. How soon after the beginning of rotation will the total acceleration vector of an arbitrary point of the body form an angle $\theta = 60°$ with its velocity vector?

## 9.5 Angular Momentum

A mass moving in a straight line has linear momentum ($\vec{P}$). When a mass rotates about some point/axis, there is momentum associated with rotational motion called the angular momentum ($\vec{L}$). Just as net external force is required to change the linear momentum of an object a net external torque is required to change the angular momentum of an object. Keeping in view the problems asked in JEE, the angular momentum is classified in following three types.

### (i) Angular momentum of a particle about some point

Suppose a particle $A$ of mass $m$ is moving with linear momentum $\vec{P} = m\vec{v}$. Its angular momentum $\vec{L}$ about point $O$ is defined as:

$$\vec{L} = \vec{r} \times \vec{P} = \vec{r} \times (m\vec{v}) = m(\vec{r} \times \vec{v})$$

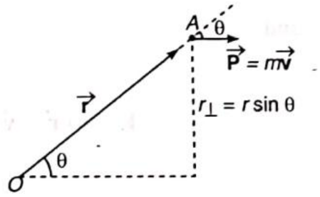



Fig. 9.31

Here, $\vec{r}$ is the radius vector of particle $A$ about $O$ at that instant of time. The magnitude of $\vec{L}$ is

$$L = mvr\sin\theta = mvr_\perp$$

Here, $r_\perp = r\sin\theta$ is the perpendicular distance of line of action of velocity $\vec{v}$ from point $O$. The direction of $\vec{L}$ is same as that of $\vec{r} \times \vec{v}$.

**Note** *The angular momentum of a particle about a line (say AB) is the component along AB of the angular momentum of the particle about any point (say O) on the line AB. This component is independent of the choice of point O, so far as it is chosen on the line AB.*

**Sample Example 9.11** *A particle of mass m is moving along the line $y = b, z = 0$ with constant speed v. State whether the angular momentum of particle about origin is increasing, decreasing or constant.*

**Solution**

$$|\vec{L}| = mvr\sin\theta$$
$$= mvr_\perp$$
$$= mvb$$

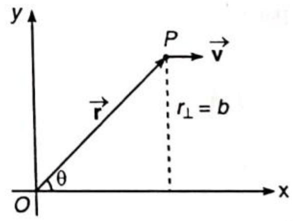



Fig. 9.32

$\therefore\ |\vec{L}| = \text{constant}$ as $m, v$ and $b$ all are constants.

Direction of $\vec{r} \times \vec{v}$ also remains the same. Therefore, angular momentum of particle about origin remains constant with due course of time.

**Note** *In this problem $|\vec{r}|$ is increasing, θ is decreasing but $r\sin\theta$, i.e., b remains constant. Hence, the angular momentum remains constant.*

**Sample Example 9.12** *A particle of mass m is projected from origin O with speed u at an angle θ with positive x-axis. Positive y-axis is in vertically upward direction. Find the angular momentum of particle at any time t about O before the particle strikes the ground again.*

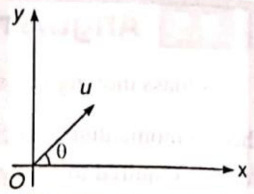


Fig. 9.33

**Solution** $$\vec{L} = m(\vec{r} \times \vec{v})$$

Here, $$\vec{r}(t) = x\hat{i} + y\hat{j} = (u\cos\theta)t\hat{i} + (ut\sin\theta - \frac{1}{2}gt^2)\hat{j}$$

and $$\vec{v}(t) = v_x\hat{i} + v_y\hat{j} = (u\cos\theta)\hat{i} + (u\sin\theta - gt)\hat{j}$$

$$\therefore \quad \vec{L} = m(\vec{r} \times \vec{v}) = m\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ (u\cos\theta)t & (u\sin\theta)t - \frac{1}{2}gt^2 & 0 \\ u\cos\theta & u\sin\theta - gt & 0 \end{vmatrix}$$

$$= m\left[(u^2\sin\theta\cos\theta)t - (u\cos\theta)gt^2 - (u^2\sin\theta\cos\theta)t + \frac{1}{2}(u\cos\theta)gt^2\right]\hat{k}$$

$$= -\frac{1}{2}m(u\cos\theta)gt^2\,\hat{k}$$

### (ii) Angular Momentum of a rigid body rotating about a fixed axis

Suppose a particle $P$ of mass $m$ is going in a circle of radius $r$ and at some instant the speed of the particle is $v$. For finding the angular momentum of the particle about the axis of rotation, the origin may be chosen anywhere on the axis. We choose it at the centre of the circle. In this case $\vec{r}$ and $\vec{P}$ are perpendicular to each other and $\vec{r} \times \vec{P}$ is along the axis. Thus, component of $\vec{r} \times \vec{P}$ along the axis is $mvr$ itself. The angular momentum of the whole rigid body about $AB$ is the sum of components of all particles, *i.e.*,

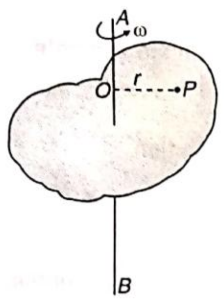


Fig. 9.34

$$L = \sum_i m_i r_i v_i$$

Here, $$v_i = r_i\omega$$

$$\therefore \quad L = \sum_i m_i r_i^2 \omega \quad \text{or} \quad L = \omega \sum_i m_i r_i^2$$

or $$L = I\omega \qquad (\text{as } \sum_i m_i r_i^2 = I)$$

Here, $I$ is the moment of inertia of the rigid body about $AB$.

**Note** *The vector relation $\vec{L} = I\vec{\omega}$ is not correct in the above case because $\vec{L}$ and $\vec{\omega}$ do not point in the same direction, but we could write $L_{AB} = I\omega$. If however the body is symmetric about the axis of rotation $\vec{L}$ and $\vec{\omega}$ are parallel and we can write ($L = I\omega$) in vector form as $\vec{L} = I\vec{\omega}$.*

*By symmetric we mean that for every mass element in the body there must be an identical mass element diametrically opposite the first element and at the same distance from the axis of rotation.*

*Thus, remember that $\vec{L} = I\vec{\omega}$ applies only to bodies that have symmetry about the (fixed) rotational axis. Here, $\vec{L}$ stands for total angular momentum. However the relation $L_{AB} = I\omega$ holds for any rigid body symmetrical or not that is rotating about a fixed axis.*

### (iii) Angular momentum of a rigid body in combined rotation and translation

Let $O$ be a fixed point in an inertial frame of reference. Angular momentum of the body about $O$ is

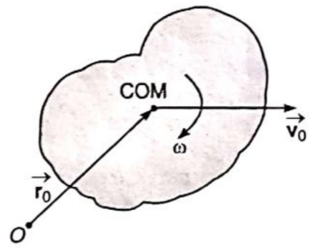


Fig. 9.35

$$\vec{L} = \sum_i m_i(\vec{r}_i \times \vec{v}_i)$$

$$= \sum_i m_i(\vec{r}_{i,cm} + \vec{r}_0) \times (\vec{v}_{i,cm} + \vec{v}_0)$$

Here, $\vec{r}_0$ is the position vector of the centre of mass and $\vec{v}_0$ its velocity.

Thus, $$\vec{L} = \sum_i m_i(\vec{r}_{i\,cm} \times \vec{v}_{i,cm}) + \{\sum_i m_i \vec{r}_{i,cm}\} \times \vec{v}_0 + \vec{r}_0 \times \{\sum_i m_i \vec{v}_{i,cm}\} + \{\sum_i m_i\}\vec{r}_0 \times \vec{v}_0$$

Now, $$\sum_i m_i \vec{r}_{i,cm} = M\vec{R}_{cm,cm} = 0$$

Similarly, $$\sum_i m_i \vec{v}_{i,cm} = M\vec{v}_{cm,cm} = 0$$

Thus, $$\vec{L} = \sum_i m_i(\vec{r}_{i,cm} \times \vec{v}_{i,cm}) + M\vec{r}_0 \times \vec{v}_0$$

or $$\vec{L} = \vec{L}_{cm} + M(\vec{r}_0 \times \vec{v}_0)$$

The first term $\vec{L}_{cm}$ represents the angular momentum of the body as seen from the centre of mass frame. The second term $M(\vec{r}_0 \times \vec{v}_0)$ equals the angular momentum of centre of mass about point $O$.

**Sample Example 9.13** *A circular disc of mass m and radius R is set into motion on a horizontal floor with a linear speed v in the forward direction and an angular speed $\omega = \dfrac{v}{R}$ in clockwise direction as shown in figure. Find the magnitude of the total angular momentum of the disc about bottommost point O of the disc.*

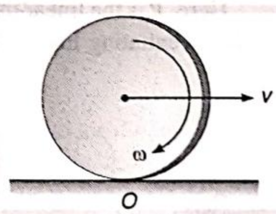


Fig. 9.36

**Solution** $$\vec{L} = \vec{L}_{cm} + m(\vec{r}_0 \times \vec{v}_0) \quad \ldots(i)$$

Here, $$\vec{L}_{cm} = I\omega \quad \text{(perpendicular to paper inwards)}$$

$$= \left(\frac{1}{2}mR^2\right)\left(\frac{v}{R}\right)$$

$$=\frac{1}{2}mvR$$

and $\quad m(\vec{r}_0 \times \vec{v}_0) = mRv \quad$ (perpendicular to paper inwards)

Since, both the terms of right hand side of Eq. (i) are in the same direction.

$$\therefore \quad |\vec{L}| = \frac{1}{2}mvR + mvR$$

or

$$|\vec{L}| = \frac{3}{2}mvR$$

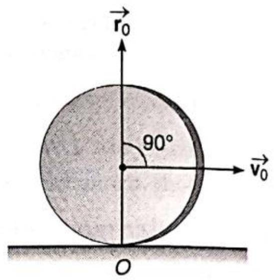


Fig. 9.37

## Introductory Exercise 9.3

1. Two particles each of mass $m$ and speed $v$, travel in opposite directions along parallel lines separated by a distance $d$. Show that the vector angular momentum of this system of particles is the same about any point taken as origin.
2. In example number 9.13 suppose the disc starts rotating anticlockwise with the same angular velocity $\omega = \frac{v}{R}$, then what will be the angular momentum of the disc about bottommost point in this new situation?
3. A particle of mass $m$ moves in $xy$ plane along the line $y = x - 4$, with constant speed $v$. Find the angular momentum of particle about origin at any instant of time $t$.
4. A particle of mass $m$ is projected from the ground with an initial speed $u$ at an angle $\alpha$. Find the magnitude of its angular momentum at the highest point of its trajectory about the point of projection.
5. If the angular momentum of a body is zero about some point. Is it necessary that it will be zero about a different point?

## 9.6 Conservation of Angular Momentum

As we have seen in Article 9.5, the angular momentum of a particle about some reference point $O$ is defined as,

$$\vec{L} = \vec{r} \times \vec{p} \qquad \text{...(i)}$$

Here, $\vec{P}$ is the linear momentum of the particle and $\vec{r}$ its position vector with respect to the reference point $O$. Differentiating Eq. (i) with respect to time, we get

$$\frac{d\vec{L}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt} + \frac{d\vec{r}}{dt} \times \vec{p} \qquad \text{...(ii)}$$

Here,

$$\frac{d\vec{p}}{dt} = \vec{F}$$

and

$$\frac{d\vec{r}}{dt} = \vec{v} \text{ (velocity of particle)}$$

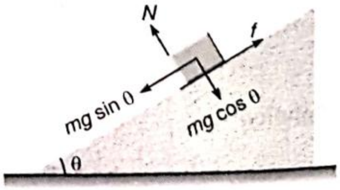


Fig. 9.38

Hence, Eq. (ii) can be rewritten as,

$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} + \vec{v} \times \vec{p}$$

Now, $\vec{\mathbf{v}} \times \vec{\mathbf{p}} = 0$, because $\vec{\mathbf{v}}$ and $\vec{\mathbf{p}}$ are parallel to each other and the cross product of two parallel vectors is zero. Thus,

$$\frac{d\vec{\mathbf{L}}}{dt} = \vec{\mathbf{r}} \times \vec{\mathbf{F}} = \vec{\tau}$$

or
$$\vec{\tau} = \frac{d\vec{\mathbf{L}}}{dt} \qquad \text{...(iii)}$$

Which states that the time rate of change of angular momentum of a particle about some reference point in an inertial frame of reference is equal to the net torques acting on it. This result is rotational analog of the equation $\vec{\mathbf{F}} = \frac{d\vec{\mathbf{P}}}{dt}$, which states that the time rate of change of the linear momentum of a particle is equal to the force acting on it. Eq. (iii) like all vector equations, is equivalent to three scalar equations, namely

$$\tau_x = \left(\frac{dL}{dt}\right)_x, \qquad \tau_y = \left(\frac{dL}{dt}\right)_y$$

and
$$\tau_z = \left(\frac{dL}{dt}\right)_z$$

The same equation can be generalised for a system of particles as, $\vec{\tau}_{\text{ext}} = \frac{d\vec{\mathbf{L}}}{dt}$. According to which the time rate of change of the total angular momentum of a system of particles about some reference point of an inertial frame of reference is equal to the sum of all external torques (of course the vector sum) acting on the system about the same reference point.

Now, suppose that $\vec{\tau}_{\text{ext}} = 0$, then $\frac{d\vec{\mathbf{L}}}{dt} = 0$, so that $\vec{\mathbf{L}} = \text{constant}$.

"When the resultant external torque acting on a system is zero, the total vector angular momentum of the system remains constant. This is the principle of the conservation of angular momentum.

For a rigid body rotating about an axis (the $z$-axis, say) that is fixed in an inertial reference frame, we have

$$L_z = I\omega$$

It is possible for the moment of inertia $I$ of a rotating body to change by rearrangement of its parts. If no net external torque acts, then $L_z$ must remain constant and if $I$ does change, there must be a compensating change in $\omega$. The principle of conservation of angular momentum in this case is expressed as

$$I\omega = \text{constant} \qquad \text{...(iv)}$$

**Sample Example 9.14** *A wheel of moment of inertia $I$ and radius $R$ is rotating about its axis at an angular speed $\omega_0$. It picks up a stationary particle of mass $m$ at its edge. Find the new angular speed of the wheel.*

**Solution** Net external torque on the system is zero. Therefore, angular momentum will remain conserved. Thus,

$$I_1\omega_1 = I_2\omega_2 \quad \text{or} \quad \omega_2 = \frac{I_1\omega_1}{I_2}$$

Here, $I_1 = I$, $\omega_1 = \omega_0$, $I_2 = I + mR^2$

$$\therefore \qquad \omega_2 = \frac{I\omega_0}{I + mR^2}$$

## Introductory Exercise 9.4

1. A thin circular ring of mass $M$ and radius $R$ is rotating about its axis with an angular speed $\omega_0$. Two particles each of mass $m$ are now attached at diametrically opposite points. Find the new angular speed of the ring.
2. If the ice at the poles melts and flows towards the equator, how will it affect the duration of day-night?
3. When tall buildings are constructed on earth, the duration of day night slightly increases. Is this statement true or false?

## 9.7 Combined Translational and Rotational Motion of a Rigid Body

Up until now we have considered only bodies rotating about some fixed axis. In JEE however questions are frequently asked on combined translational and rotational motion of a rigid body.

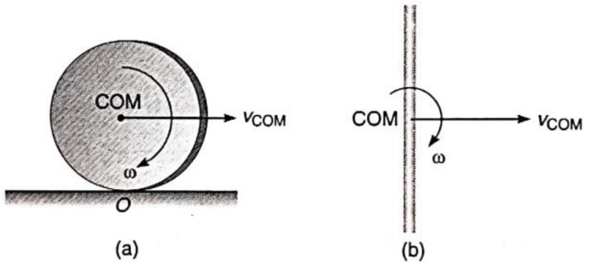



Fig. 9.39

In such problems if two things are known:
(i) velocity of centre of mass ($v_{COM}$) (ii) angular velocity of the rigid body ($\omega$).

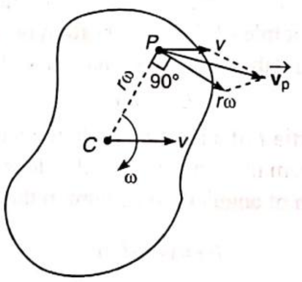



Fig. 9.40

The motion of whole rigid body can be described.

For example, let the velocity of centre of mass of a rigid body shown in figure is $v$ and angular velocity of the rigid body is $\omega$. Then velocity of any point $P$ on the rigid body can be obtained as,

$$\vec{\mathbf{v}}_p = \vec{\mathbf{v}}_{COM} + \vec{\mathbf{v}}_{p,COM}$$

Here, $$|\vec{\mathbf{v}}_{COM}| = v$$

and $\vec{\mathbf{v}}_{p,COM} = r\omega$ in a direction perpendicular to line $CP$.

Thus, the velocity of point $P$ is the vector sum of $\vec{\mathbf{v}}_{COM}$ and $\vec{\mathbf{v}}_{p,COM}$ as shown in figure.

### Kinetic energy of rigid body in combined translational and rotational motion

Here, two energies are associated with the rigid body. One is translational $\left(=\frac{1}{2} m v_{COM}^2\right)$ and another is rotational $\left(=\frac{1}{2} I_{COM}\ \omega^2\right)$. Thus, total kinetic energy of the rigid body is

$$K = \frac{1}{2} m v_{COM}^2 + \frac{1}{2} I_{COM}\ \omega^2$$

For better understanding of this article let us take an example based on the above theory asked in JEE-2000.

**Sample Example 9.15** *A disc of radius R has linear velocity v and angular velocity ω as shown in the figure. Given v = rω. Find velocity of points A, B, C and D on the disc.*

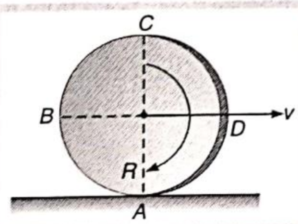



Fig. 9.41

**Solution** As stated in above article, velocity of any point of the rigid body in rotation plus translation is the vector sum of $v$ (the velocity of centre of mass) and $r\omega$. Here, $r$ is the distance of the point under consideration from the centre of mass of the body. Direction of this $r\omega$ is perpendicular to the line joining the point with centre of mass in the sense of rotation. Based on this, velocities of points $A$, $B$, $C$ and $D$ are as shown below:

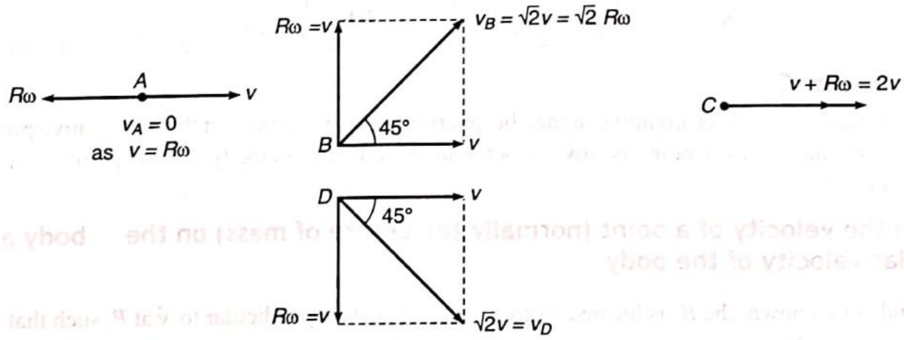



Fig. 9.42

Thus, $v_A$ is zero, velocity of $B$ and $D$ is $\sqrt{2}v$ or $\sqrt{2}R\omega$ and velocity of $C$ is $2v$ or $2R\omega$ in the directions shown in figure.

## 9.8 Instantaneous Axis of Rotation

The combined effects of translation of the centre of mass and rotation about an axis through the centre of mass are equivalent to a pure rotation with the same angular speed about an axis passing through a point of zero velocity. Such an axis is called the instantaneous axis of rotation. (*IAOR*). This axis is always perpendicular to the plane used to represent the motion and the intersection of the axis with this plane defines the location of instantaneous centre of zero velocity (*IC*).

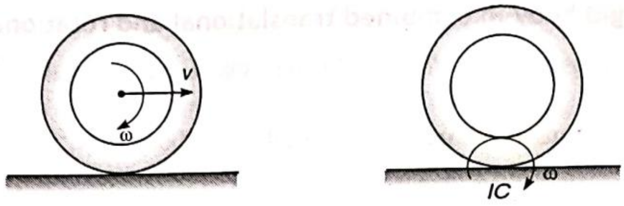



Fig. 9.43

For example consider a wheel which rolls without slipping. In this case the point of contact with the ground has zero velocity. Hence, this point represents the *IC* for the wheel. If it is imagined that the wheel is momentarily pinned at this point, the velocity of any point on the wheel can be found using $v = r\omega$. Here $r$ is the distance of the point from *IC*. Similarly, the kinetic energy of the body can be assumed to be pure rotational about *IAOR* or,

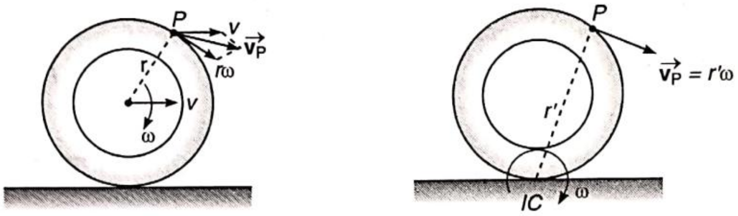



Fig. 9.44

$$K = \frac{1}{2} I_{IAOR}\ \omega^2$$

Rotation + Translation $\Rightarrow$ Pure rotation about *IAOR* passing through *IC*

$$\text{KE} = \frac{1}{2} mv_{\text{COM}}^2 + \frac{1}{2} I_{\text{COM}}\omega^2 \quad \Rightarrow \quad \text{KE} = \frac{1}{2} I_{IAOR}\ \omega^2$$

## Location of the IC

If the location of the *IC* is unknown, it may be determined by using the fact that the relative position vector extending from the *IC* to a point is always perpendicular to the velocity of the point. Following three possibilities exist.

### (i) Given the velocity of a point (normally the centre of mass) on the body and the angular velocity of the body

If $v$ and $\omega$ are known, the *IC* is located along the line drawn perpendicular to $\vec{v}$ at *P*, such that the distance from *P* to *IC* is, $r = \frac{v}{\omega}$. Note that *IC* lie on that side of *P* which causes rotation about the *IC*, which is consistent with the direction of motion caused by $\vec{\omega}$ and $\vec{v}$.

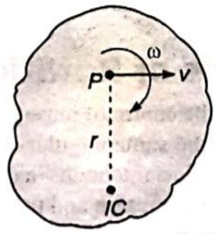



Fig. 9.45

**Sample Example 9.16** *A rotating disc moves in the positive direction of the x-axis. Find the equation $y(x)$ describing the position of the instantaneous axis of rotation if at the initial moment the centre c of the disc was located at the point O after which it moved with constant velocity v while the disc started rotating counterclockwise with a constant angular acceleration $\alpha$. The initial angular velocity is equal to zero.*

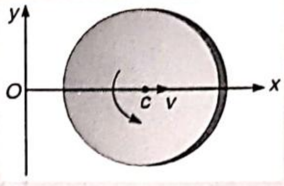



Fig. 9.46

**Solution**

$$t = \frac{x}{v}$$

and

$$\omega = \alpha t = \frac{\alpha x}{v}$$

The position of *IAOR* will be at a distance

$$y = \frac{v}{\omega}$$

or

$$y = \frac{v}{\frac{\alpha x}{v}}$$

or

$$y = \frac{v^2}{\alpha x}$$

or

$$xy = \frac{v^2}{\alpha} = \text{constant}$$

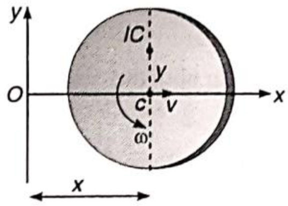



Fig. 9.47

This is the desired *x-y* equation. This equation represents a rectangular hyperbola.

### (ii) Given the lines of action of two non-parallel velocities

Consider the body shown in figure where the line of action of the velocities $\vec{v}_A$ and $\vec{v}_B$ are known. Draw perpendiculars at *A* and *B* to these lines of action. The point of intersection of these perpendiculars as shown locates the *IC* at the instant considered.

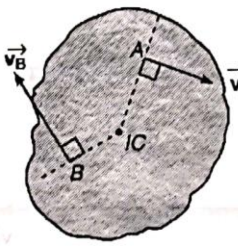



Fig. 9.48

### (iii) Given the magnitude and direction of two parallel velocities

When the velocities of points *A* and *B* are parallel and have known magnitudes $v_A$ and $v_B$ then the location of the *IC* is determined by proportional triangles as shown in figure.

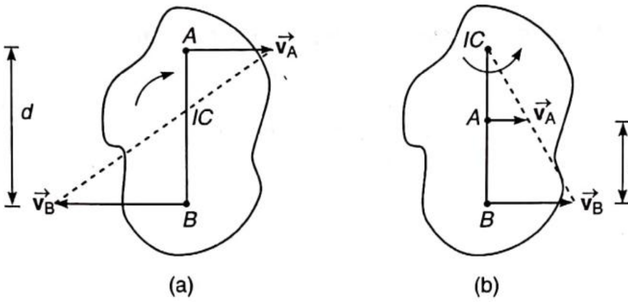



Fig. 9.49

In both the cases, $$r_{A,IC} = \frac{v_A}{\omega}$$

and $$r_{B,IC} = \frac{v_B}{\omega}$$

In Fig. (a) $$r_{A,IC} + r_{B,IC} = d$$

and in Fig. (b) $$r_{B,IC} - r_{A,IC} = d$$

As a special case, if the body is translating, $v_A = v_B$ and the $IC$ would be located at infinity, in which case $\omega = 0$.

**Sample Example 9.17** *A uniform thin rod of mass m and length l is standing on a smooth horizontal surface. A slight disturbance causes the lower end to slip on the smooth surface and the rod starts falling. Find the velocity of centre of mass of the rod at the instant when it makes an angle θ with horizontal.*

**Solution** As the floor is smooth, mechanical energy of the rod will remain conserved. Further, no horizontal force acts on the rod, hence the centre of mass moves vertically downwards in a straight line. Thus velocities of $COM$ and the lower end $B$ are in the directions shown in figure. The location of $IC$ at this instant can be found by drawing perpendiculars to $\vec{v}_C$ and $\vec{v}_B$ at respective points. Now, the rod may be assumed to be in pure rotational motion about $IAOR$ passing through $IC$ with angular speed $\omega$.

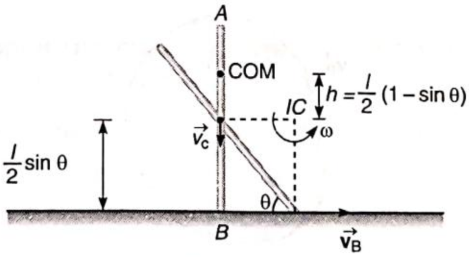



Fig. 9.50

Applying conservation of mechanical energy. Decrease in gravitational potential energy of the rod = increase in rotational kinetic energy about $IAOR$

$$\therefore \quad mgh = \frac{1}{2} I_{IAOR}\, \omega^2$$

or $$mg\frac{l}{2}(1 - \sin\theta) = \frac{1}{2}\left(\frac{ml^2}{12} + \frac{ml^2}{4}\cos^2\theta\right)\omega^2$$

Solving this equation, we get

$$\omega = \sqrt{\frac{12g(1-\sin\theta)}{l(1+3\cos^2\theta)}}$$

Now,

$$|\vec{v}_C| = \left(\frac{l}{2}\cos\theta\right)\omega$$

$$= \sqrt{\frac{3gl(1-\sin\theta)\cos^2\theta}{(1+3\cos^2\theta)}}$$

## Introductory Exercise 9.5

**1.** In sample example 9.16, find the equation $y(x)$ if at the initial moment the axis $c$ of the disc was located at the point $O$ after which it moved with a constant linear acceleration $a_0$ (and the zero initial velocity) while the disc rotates counter clockwise with a constant angular velocity $\omega$.

**2.** A uniform bar of length $l$ stands vertically touching a wall $OA$. When slightly displaced, its lower end begins to slide along the floor. Obtain an expression for the angular velocity $\omega$ of the bar as a function of $\theta$. Neglect friction everywhere.

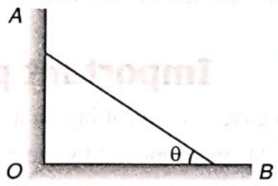



Fig. 9.51

## 9.9 Uniform Pure Rolling

Pure rolling means no relative motion (or no slipping) at point of contact between two bodies.

For example, consider a disc of radius $R$ moving with linear velocity $v$ and angular velocity $\omega$ on a horizontal ground. The disc is said to be moving without slipping if velocities of points $P$ and $Q$ (shown in figure b) are equal, *i.e.*,

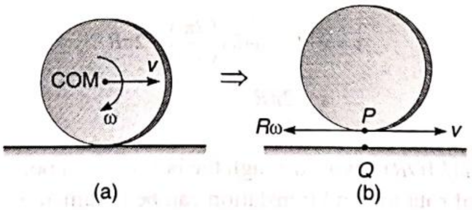



Fig. 9.52

$$v_P = v_Q$$

or $$v - R\omega = 0$$

or $$v = R\omega$$

If $v_P > v_Q$ or $v > R\omega$, the motion is said to be forward slipping and if $v_P < v_Q$ or $v < R\omega$, the motion is said the backward slipping (or sometimes called forward english).

Thus, $v = R\omega$ is the condition of pure rolling on a stationary ground. Sometimes it is simply said rolling. Suppose the base over which the disc in rolling, is also moving with some velocity (say $v_0$) then in that case condition of pure rolling is different.

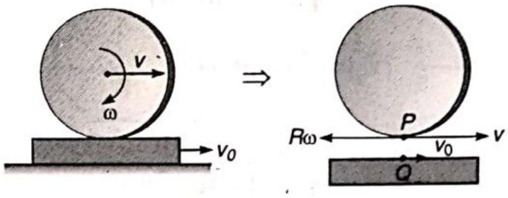



Fig. 9.53

For example, in the above figure,

$$v_P = v_Q$$

or
$$v - R\omega = v_0$$

Thus, in this case $v - R\omega \neq 0$, but $v - R\omega = v_0$. By uniform pure rolling we mean that $v$ and $\omega$ are constant. They are neither increasing nor decreasing.

## Important points in Uniform Pure Rolling On Ground

In case of pure rolling on a stationary horizontal ground, following points are important to note:

- Distance moved by the centre of mass of the rigid body in one full rotation is $2\pi R$.

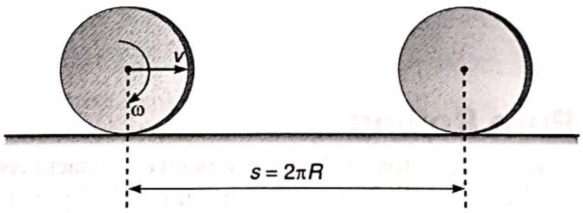



Fig. 9.54

This can be shown as under:

$$s = v \cdot T = (\omega R)\left(\frac{2\pi}{\omega}\right) = 2\pi R$$

In forward slipping $\quad s > 2\pi R \quad$ (as $v > \omega R$)
and in backward slipping $\quad s < 2\pi R \quad$ (as $v < \omega R$)

- Instantaneous axis of rotation ($IAOR$) passes through the bottommost point, as it is a point of zero velocity. Thus, the combined motion of rotation and translation can be assumed to be pure rotational motion about bottommost point with same angular speed $\omega$.

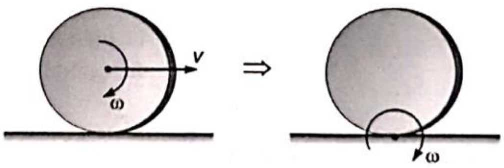



Fig. 9.55

- The speed of a point on the circumference of the body at the instant shown in figure is $2v\sin\frac{\theta}{2}$ or $2R\omega\sin\frac{\theta}{2}$. *i.e.*,

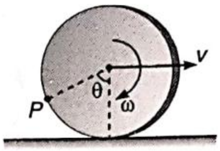


Fig. 9.56

$$|\vec{v}_P| = v_p = 2v\sin\frac{\theta}{2} = 2R\omega\sin\frac{\theta}{2}$$

This can be shown by following two methods.

**Method 1:**

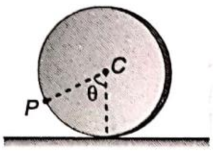


Fig. 9.57

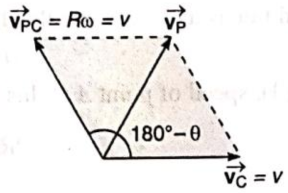


Fig. 9.58

$$\vec{v}_P = \vec{v}_C + \vec{v}_{PC}$$

$\therefore$
$$|\vec{v}_P| = \sqrt{v^2 + v^2 + 2v\cdot v\cos(180° - \theta)}$$
$$= 2v\sin\frac{\theta}{2}$$

**Method 2:**

$$|\vec{v}_P| = (OP)\omega$$

Here,
$$OP = 2R\sin\frac{\theta}{2}$$

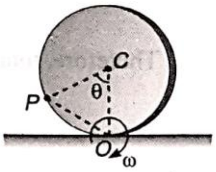


Fig. 9.59

$\therefore$
$$|\vec{v}_P| = \left(2R\sin\frac{\theta}{2}\right)\omega$$
$$= 2R\omega\sin\left(\frac{\theta}{2}\right)$$
$$= 2v\sin\frac{\theta}{2}$$

- From point number (3) we can see that

$$v_A = 0 \quad \text{as} \quad \theta = 0°$$
$$v_B = \sqrt{2}v \quad \text{as} \quad \theta = 90°$$

and
$$v_C = 2v \quad \text{as} \quad \theta = 180°$$

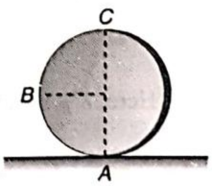


Fig. 9.60

- The path of a point on circumference is a cycloid and the distance moved by this point in one full rotation is $8R$.

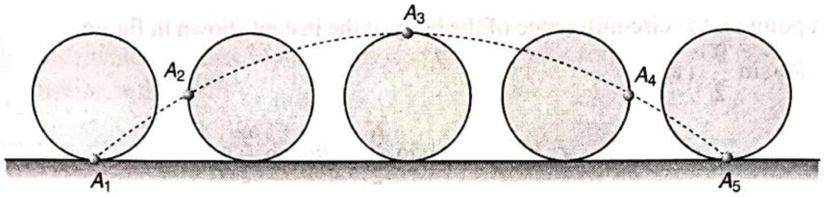



**Fig. 9.61**

In the figure, the dotted line is a cycloid and the distance $A_1 A_2 \dots A_5$ is $8R$. This can be proved as under.

In figure 9.61 $\qquad \theta = \omega t$

According to point (3), speed of point $A$ at this moment is,

$$v_A = 2R\omega \sin\left(\frac{\omega t}{2}\right)$$

Distance moved by it in time $dt$ is,

$$ds = v_A \; dt = 2R\omega \sin\left(\frac{\omega t}{2}\right) dt$$

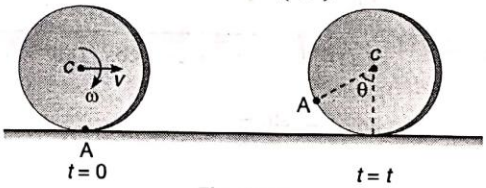



**Fig. 9.62**

Therefore, total distance moved in one full rotation is,

$$s = \int_0^{T = 2\pi/\omega} ds$$

or

$$s = \int_0^{T=2\pi/\omega} 2R\omega \sin\left(\frac{\omega t}{2}\right) dt$$

On integration we get, $\qquad s = 8R.$

$$\frac{K_R}{K_T} = 1 \text{ for a ring} = \frac{1}{2} \text{ for a disc}$$

$$= \frac{2}{5} \text{ for a solid sphere}$$

$$= \frac{2}{3} \text{ for a hollow sphere etc.}$$

Here, $K_R$ stands for rotational kinetic energy $\left(= \frac{1}{2} I\omega^2\right)$ and $K_T$ for translational kinetic energy $\left(= \frac{1}{2} mv^2\right)$. For example, for a disc :

$$K_R = \frac{1}{2} I\omega^2 = \frac{1}{2}\left(\frac{1}{2} mR^2\right)\left(\frac{v}{R}\right)^2 = \frac{1}{4} mv^2 \quad \text{and} \quad K_T = \frac{1}{2} mv^2$$

$$\therefore \qquad \frac{K_R}{K_T} = \frac{1}{2}$$

**Sample Example 9.18** *A disc of radius R start at time $t = 0$ moving along the positive x axis with linear speed v and angular speed $\omega$. Find the x and y coordinates of the bottommost point at any time t.*

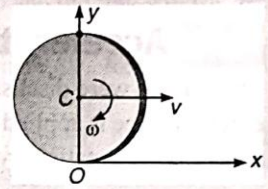


Fig. 9.63

**Solution** At time $t$ the bottommost point will rotate an angle $\theta = \omega t$ with respect to the centre of the disc $C$. The centre $C$ will travel a distance $s = vt$.

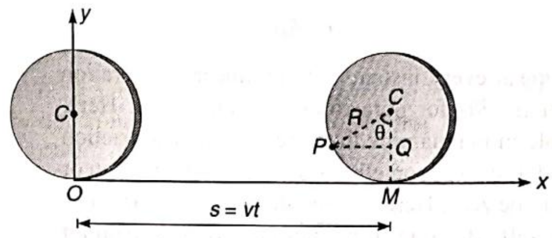


Fig. 9.64

In the figure, $PQ = R \sin \theta = R \sin \omega t$

and $CQ = R \cos \theta = R \cos \omega t$

Coordinates of point $P$ at time $t$ are,

$$x = OM - PQ = vt - R \sin \omega t$$

and $$y = CM - CQ = R - R \cos \omega t$$

$\therefore$ $$(x, y) \equiv (vt - R \sin \omega t, R - R \cos \omega t)$$

## Introductory Exercise 9.6

**1.** A solid sphere of mass $m$ rolls down an inclined plane a height $h$. Find rotational kinetic energy of the sphere.

[**Hint :** Mechanical energy will remain conserved]

**2.** A ring of radius $R$ rolls on a horizontal ground with linear speed $v$ and angular speed $\omega$. For what value of $\theta$ the velocity of point $P$ is in vertical direction. ($v < R\omega$)

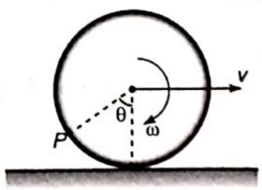


Fig. 9.65

**3.** The topmost and bottommost velocities of a disc are $v_1$ and $v_2$ ($< v_1$) in the same direction. The radius is $R$. Find the value of angular velocity $\omega$.

## 9.10 Accelerated Pure Rolling

So, far we were discussing the uniform pure rolling in which $v$ and $\omega$ were constants. Now, suppose an external force is applied to the rigid body, the motion will no longer remain uniform. The condition of pure rolling on a stationary ground is,

$$v = R\omega$$

Differentiating this equation with respect to time, we have

$$\frac{dv}{dt} = R \cdot \frac{d\omega}{dt}$$

or

$$a = R\alpha$$

Thus, in addition to $v = R\omega$ at every instant of time, linear acceleration $= R \times$ angular acceleration or $a = R\alpha$ for pure rolling to take place. Here, friction plays an important role in maintaining the pure rolling. The friction may sometimes act in forward direction, sometimes in backward direction or under certain conditions it may be zero. Here, we should not forget the basic nature of friction, which is a self adjusting force (upto a certain maximum limit) and which has a tendency to stop the relative motion between two bodies in contact. Let us take an example illustrating the above theory.

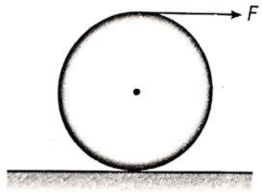



Fig. 9.66

Suppose a force $F$ is applied at the topmost point of a rigid body of radius $R$, mass $M$ and moment of inertia $I$ about an axis passing through the centre of mass. Now, the applied force $F$ can produce by itself:

(i) a linear acceleration $a$ and

(ii) an angular acceleration $\alpha$.

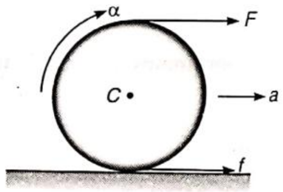



Fig. 9.67

If $a = R\alpha$, then there is no need of friction and force of friction $f = 0$. If $a < R\alpha$, then to support the linear motion the force of friction $f$ will act in forward direction. Similarly, if $a > R\alpha$, then to support the angular motion the force of friction will act in backward direction. So, in this case force of friction will be either backward, forward or even zero also. It all depends on $M$, $I$ and $R$. For calculation you can choose any direction of friction. Let we assume it in forward direction,

Let, $a =$ linear acceleration, $\alpha =$ angular acceleration

then,
$$a = \frac{F_{net}}{M} = \frac{F + f}{M} \quad \text{...(i)}$$

$$\alpha = \frac{\tau_c}{I} = \frac{(F - f)R}{I} \quad \text{...(ii)}$$

For pure rolling to take place,

$$a = R\alpha \quad \text{...(iii)}$$

Solving Eqs. (i), (ii) and (iii), we get

$$f = \frac{(MR^2 - I)}{(MR^2 + I)} \cdot F \quad \text{...(iv)}$$

From Eq. (iv) following conclusions can be drawn

(i) If $I = MR^2$ (e.g., in case of a ring)

$$f = 0$$

*i.e.*, if a force $F$ is applied on the top of a ring, the force of friction will be zero and the ring will roll without slipping.

(ii) If $I < MR^2$, (*e.g.*, in case of a solid sphere or a hollow sphere), $f$ is positive, *i.e.*, force of friction will be forward.

(iii) If $I > MR^2$, $f$ is negative, *i.e.*, force of friction will be backwards. Although under no condition $I > MR^2$. (Think why?). So force of friction is either in forward direction or zero.

Here, it should be noted that the force of friction $f$ obtained in Eq. (iv) should be less than the limiting friction ($\mu Mg$), for pure rolling to take place. Further, we saw that if $I < MR^2$ force of friction acts in forward direction. This is because $\alpha$ is more if $I$ is small $\left(\alpha = \frac{\tau}{I}\right)$ *i.e.*, to support the linear motion force of friction is in forward direction.

**Note** *It is often said that rolling friction is less than the sliding friction. This is because the force of friction calculated by equation number (iv) normally comes less than the sliding friction ($\mu_k N$) and even sometimes it is in forward direction, i.e., it supports the motion.*

There are certain situations in which the direction of friction is fixed. For example in the following situations the force of friction is backward.

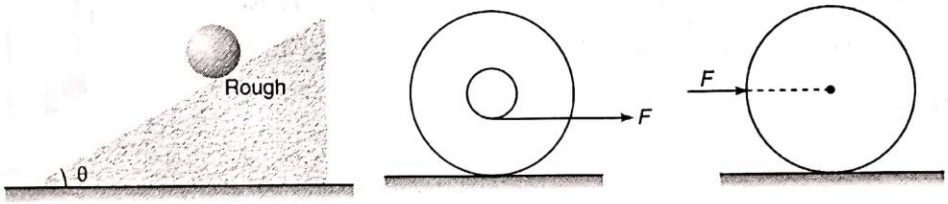



Fig. 9.68

## Rolling on Rough Inclined Plane

As we said earlier also, force of friction in this case will be backward. Equations of motion are :

$$a = \frac{Mg\sin\theta - f}{M} \quad \text{...(i)}$$

$$\alpha = \frac{fR}{I} \quad \text{...(ii)}$$

For pure rolling to take place,

$$a = R\alpha \quad \text{...(iii)}$$

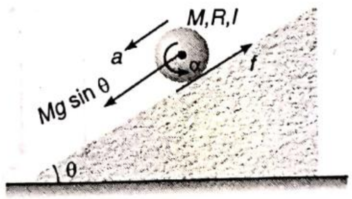



Fig. 9.69

Solving Eqs. (i), (ii) and (iii), we get

$$f = \frac{Mg\sin\theta}{1 + \frac{MR^2}{I}} \quad \text{...(iv)}$$

and

$$a = \frac{g\sin\theta}{1 + \frac{I}{MR^2}} \quad \text{...(v)}$$

From Eq. (v), we can see that if a solid sphere and a hollow sphere of same mass and radius are released from a rough inclined plane the solid sphere reaches the bottom first because :

$$I_{\text{solid}} < I_{\text{hollow}} \quad \text{or} \quad a_{\text{solid}} > a_{\text{hollow}}$$

$$\therefore \qquad t_{\text{solid}} < t_{\text{hollow}}$$

Further, the force of friction calculated in Eq. (iv) for pure rolling to take place should be less than or equal to the maximum friction $\mu Mg \cos\theta$.

or
$$\frac{Mg \sin\theta}{1+\dfrac{MR^2}{I}} \le \mu\, Mg \cos\theta$$

or
$$\mu \ge \frac{\tan\theta}{1+\dfrac{MR^2}{I}}$$

**Sample Example 9.19** *In the arrangement shown in figure the mass of the uniform solid cylinder of radius R is equal to m and the masses of two bodies are equal to $m_1$ and $m_2$. The thread slipping and the friction in the axle of the cylinder are supposed to be absent. Find the angular acceleration of the cylinder and the ratio of tensions $\frac{T_1}{T_2}$ of the vertical sections of the thread in the process of motion.*

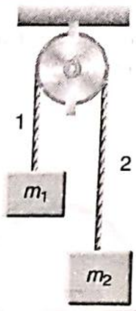



Fig. 9.70

**Solution** Let $\alpha$ = angular acceleration of the cylinder

and $a$ = linear acceleration of two bodies

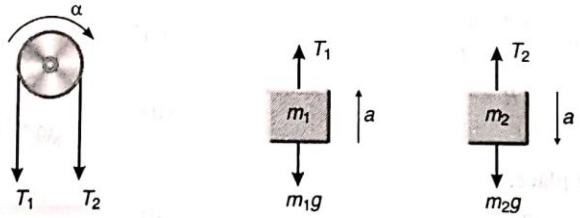



Fig. 9.71

Equations of motion are:

For mass $m_1$, $\qquad T_1 - m_1 g = m_1 a \qquad$ ...(i)

For mass $m_2$, $\qquad m_2 g - T_2 = m_2 a \qquad$ ...(ii)

For cylinder, $\qquad \alpha = \dfrac{(T_2 - T_1)R}{\frac{1}{2} mR^2} \qquad$ ...(iii)

For no slipping condition $\qquad a = R\alpha \qquad$ ...(iv)

Solving these equations, we get

$$\alpha = \frac{2(m_2 - m_1)g}{(2m_1 + 2m_2 + m)R}$$

and

$$\frac{T_1}{T_2} = \frac{m_1(m + 4m_2)}{m_2(m + 4m_1)}$$

**Sample Example 9.20** *Consider the arrangement shown in figure. The string is wrapped around a uniform cylinder which rolls without slipping. The other end of the string is passed over a massless, frictionless pulley to a falling weight. Determine the acceleration of the falling mass m in terms of only the mass of the cylinder M, the mass m and g.*

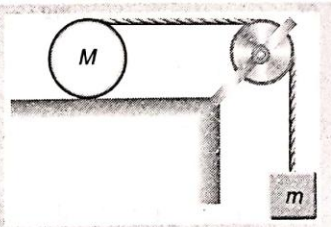



**Fig. 9.72**

**Solution** Let $T$ be the tension in the string and $f$ the force of (static) friction, between the cylinder and the surface

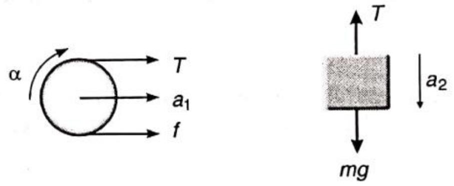



**Fig. 9.73**

$a_1$ = acceleration of centre of mass of cylinder towards right
$a_2$ = downward acceleration of block $m$
$\alpha$ = angular acceleration of cylinder (clockwise)

Equations of motion are:

For block, $$mg - T = ma_2 \quad \text{...(i)}$$

For cylinder, $$T + f = Ma_1 \quad \text{...(ii)}$$

$$\alpha = \frac{(T - f)R}{\frac{1}{2}MR^2} \quad \text{...(iii)}$$

The string attaches the mass $m$ to the highest point of the cylinder, hence

$$v_m = v_{COM} + R\omega$$

Differentiating, we get $$a_2 = a_1 + R\alpha \quad \text{...(iv)}$$

We also have (for rolling without slipping)

$$a_1 = R\alpha \quad \text{...(v)}$$

Solving these equations, we get $$a_2 = \frac{8mg}{3M + 8m}$$

## Alternate Solution (Energy Method)

Since, there is no slipping at all contacts mechanical energy of the system will remain conserved.

∴ Decrease in gravitational potential energy of block $m$ in time $t$ = increase in translational kinetic energy of block + increase in rotational as well as translational kinetic energy of cylinder.

∴ $$mgh = \frac{1}{2} mv_2^2 + \frac{1}{2} I\omega^2 + \frac{1}{2} Mv_1^2$$

or $$mg\left(\frac{1}{2} a_2 t^2\right) = \frac{1}{2} m (a_2 t)^2 + \frac{1}{2}\left(\frac{1}{2} MR^2\right)(\alpha t)^2 + \frac{1}{2} M(a_1 t)^2 \quad \ldots\text{(vi)}$$

Solving Eqs. (iv), (v) and (vi), we get the same result.

## Introductory Exercise 9.7

1. A ball of mass $M$ and radius $R$ is released on a rough inclined plane of inclination $\theta$. Friction is not sufficient to prevent slipping. The coefficient of friction between the ball and the plane is $\mu$. Find:
   (a) the linear acceleration of the ball down the plane,
   (b) the angular acceleration of the ball about its centre of mass.
2. Work done by friction in pure rolling is always zero. Is this statement true or false?
3. A spool is pulled by a force in vertical direction as shown in figure. What is the direction of friction in this case? The spool does not loose contact with the ground.

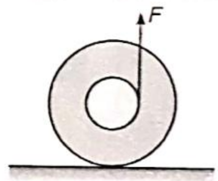



Fig. 9.74

4. A cylinder is rolling down a rough inclined plane. Its angular momentum about the point of contact remains constant. Is this statement true or false?
5. Two forces $F_1$ and $F_2$ are applied on a spool of mass $M$ and moment of inertia $I$ about an axis passing through its centre of mass. Find the ratio $\frac{F_1}{F_2}$, so that the force of friction is zero. Given that $I < 2Mr^2$.

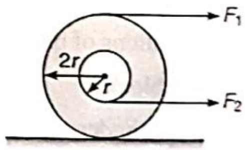



Fig. 9.75

6. A disc is placed on the ground. Friction coefficient is $\mu$. What is the minimum force required to move the disc if it is applied at the topmost point?
7. When a body rolls, on a stationary ground, the acceleration of the point of contact is always zero. Is this statement true or false?

## 9.11 Angular Impulse

The angular impulse of a torque in a given time interval is defined as $\int_{t_1}^{t_2} \vec{\tau}\, dt$

Here, $\vec{\tau}$ is the resultant torque acting on the body. Further, since

$$\vec{\tau} = \frac{d\vec{\mathbf{L}}}{dt} \qquad \therefore \qquad \vec{\tau}\, dt = d\vec{\mathbf{L}}$$

or
$$\int_{t_1}^{t_2} \vec{\tau} dt = \text{angular impulse} = \vec{\mathbf{L}}_2 - \vec{\mathbf{L}}_1$$

Thus, the angular impulse of the resultant torque is equal to the change in angular momentum. Let us take few examples based on the angular impulse.

**Sample Example 9.21** *A uniform sphere of mass m and radius R starts rolling without slipping down an inclined plane. Find the time dependence of the angular momentum of the sphere relative to the point of contact at the initial moment. How will the result be affected in the case of a perfectly smooth inclined plane? The angle of inclination of the plane is θ.*

**Solution** Applying the equation

Angular impulse = change in angular momentum about point of contact we have,

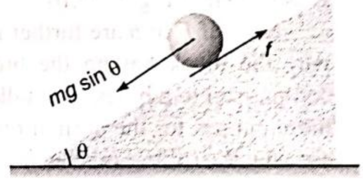


Fig. 9.76

$$\int \vec{\tau}\, dt = \Delta\vec{\mathbf{L}}$$

or
$$L = (mg \sin\theta)\, Rt$$

There will be no change in the result, as the torque of force of friction in the first case is zero about point of contact. So, it hardly matters whether the surface is rough or smooth.

## 9.12 Toppling

You might have seen in your practical life that if a force $F$ is applied to a block $A$ of smaller width it is more likely to topple down, before sliding while if the same force $F$ is applied to an another block $B$ of broader base, chances of its sliding are more compared to its toppling. Have you ever throught why it happens so. To understand it better let us take an example.

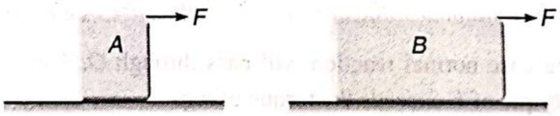


Fig. 9.77

Suppose a force $F$ is applied at a height $b$ above the base $AE$ of the block. Further, suppose the friction $f$ is sufficient to prevent sliding. In this case, if the normal reaction $N$ also passes through $C$, then despite the fact that the block is in translational equilibrium ($F = f$ and $N = mg$), an unbalanced torque (due to the couple of forces $F$ and $f$) is there. This torque has a tendency to topple the block about point $E$. To cancel the effect of this unbalanced torque the normal reaction $N$ is shifted towards right a distance '$a$' such that, net anticlockwise torque is equal to the net clockwise torque or

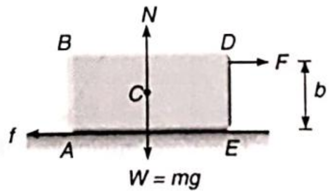


Fig. 9.78

$$Fb = (mg)a$$

or $$a = \frac{Fb}{mg}$$

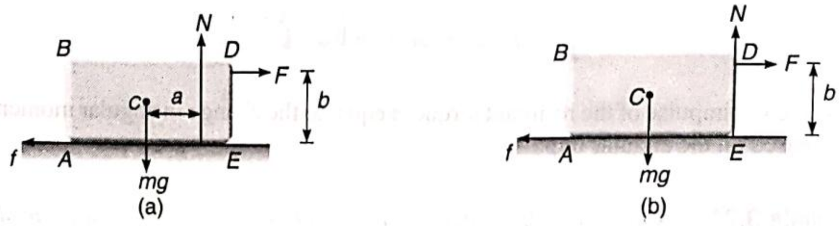



**Fig. 9.79**

Now, as $F$ or $b$ (or both) are increased, distance $a$ also increases. But it can not go beyond the right edge of the block. So, in extreme case (beyond which the block will topple down), the normal reaction passes through $E$ as shown in Fig. 9.79(b).

Now, if $F$ or $b$ are further increased, the block will topple down. This is why the block having the broader base has less chances of toppling in comparison to a block of smaller base. Because the block of larger base has more margin for the normal reaction to shift. On the similar ground we can see why the rolling is so easy.

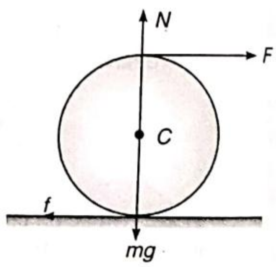



**Fig. 9.80**

Because in this case the normal reaction has zero margin to shift. So even if the body is in translational equilibrium ($F = f, N = mg$) an unbalanced torque is left behind and the body starts rolling clockwise. As soon as the body starts rolling the force of friction is so adjusted (both in magnitude and direction) that either the pure rolling starts (if friction is sufficient enough) or the body starts sliding. Let us take few examples related to toppling.

**Sample Example 9.22** *A uniform cube of side a and mass m rests on a rough horizontal table. A horizontal force F is applied normal to one of the faces at a point directly above the centre of the face, at a height* $\frac{3a}{4}$ *above the base. What is the minimum value of F for which the cube begins to tip about an edge?*

**Solution** In the limiting case normal reaction will pass through $O$. The cube will tip about $O$ if torque of $F$ exceeds the torque of $mg$.

Hence, $$F\left(\frac{3a}{4}\right) > mg\left(\frac{a}{2}\right)$$

or $$F > \frac{2}{3} mg$$

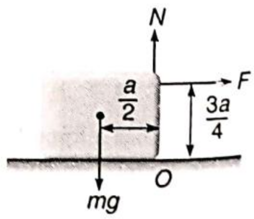



**Fig. 9.81**

Therefore, minimum value of $F$ is $\frac{2}{3} mg$.

**Sample Example 9.23** *A uniform cylinder of height h and radius r is placed with its circular face on a rough inclined plane and the inclination of the plane to the horizontal is gradually increased. If μ is the coefficient of friction, then under what conditions the cylinder will (a) slide before toppling (b) topple before sliding.*

**Solution** (a) The cylinder will slide if

$$mg \sin \theta > \mu mg \cos \theta$$

or $\tan \theta > \mu$ ...(i)

The cylinder will topple if $(mg \sin \theta)\frac{h}{2} > (mg \cos \theta) r$

or $\tan \theta > \frac{2r}{h}$ ...(ii)

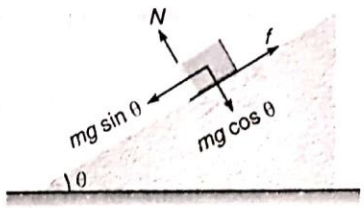



Fig. 9.82

Thus, the condition of sliding is $\tan \theta > \mu$ and condition of toppling is $\tan \theta > \frac{2r}{h}$. Hence, the cylinder will slide before toppling if

$$\mu < \frac{2r}{h}$$

(b) The cylinder will topple before sliding if $\mu > \frac{2r}{h}$

## Introductory Exercise 9.8

**1.** A cube is resting on an inclined plane. If the angle of inclination is gradually increased, what must be the coefficient of friction between the cube and plane so that,
(a) cube slides before toppling?
(b) cube topples before sliding?

**2.** A solid sphere of mass $M$ and radius $R$ is hit by a cue at a height $h$ above the centre $C$. For what value of $h$ the sphere will roll without slipping?

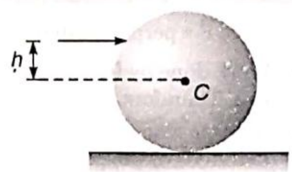



Fig. 9.83

## Extra Points

- Whether a particle is in translational motion, rotational motion or in both it merely depends on the reference point with respect to which the motion of the particle is described.

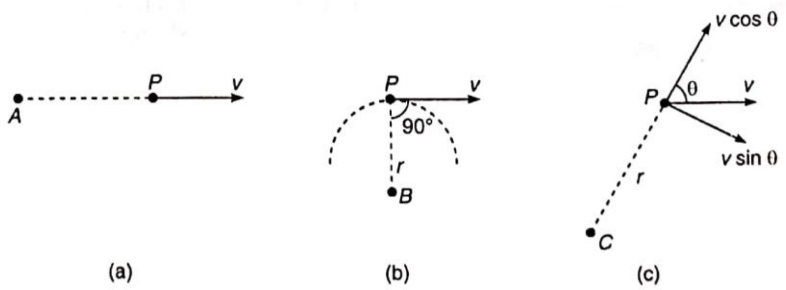



Fig. 9.84

**For example :** Suppose a particle $P$ of mass $m$ is moving in a straight line as shown in figures (a), (b) and (c).

**Refer figure (a) :** With respect to point $A$, the particle is in pure translational motion. Hence, kinetic energy of the particle can be written as

$$KE = \frac{1}{2} mv^2$$

**Refer figure (b) :** With respect to point $B$, the particle is in pure rotational motion. Hence, the kinetic energy of the particle can be written as

$$KE = \frac{1}{2} I\omega^2 = \frac{1}{2}(mr^2)\left(\frac{v}{r}\right)^2$$
$$= \frac{1}{2} mv^2$$

**Refer figure (c) :** With respect to point $C$, the particle can be assumed to be in rotational as well as translational motion. Hence, the kinetic energy of the particle can be written as

$$KE = \frac{1}{2} m(v\cos\theta)^2 + \frac{1}{2} I\omega^2$$
$$= \frac{1}{2} m(v\cos\theta)^2 + \frac{1}{2}(mr^2)\left(\frac{v\sin\theta}{r}\right)^2$$
$$= \frac{1}{2} mv^2.$$

Thus, in all the three cases, the kinetic energy of the particle comes out to be the same.

- If $\vec{\mathbf{F}}$ is perpendicular to $AB$, but does not intersect it, then component of torque about line $AB$ = magnitude of force $\vec{\mathbf{F}}$ × perpendicular distance of $\vec{\mathbf{F}}$ from the line $AB$ (called the lever arm or moment arm) of this torque.
- Work done by friction in pure rolling on a stationary ground in zero as the point of application of the force is at rest. Therefore, mechanical energy can be conserved if all other dissipative forces are ignored.
- In cases where pulley is having some mass and friction is sufficient enough to prevent slipping, the tension on two sides of the pulley will be different and rotational motion of the pulley is also to be considered.
-

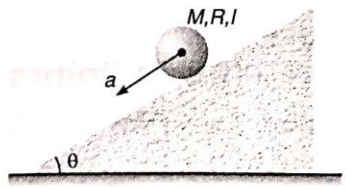



Fig. 9.85

| | |
|---|---|
| $a = g\sin\theta$ | if surface is smooth |
| $a = g\sin\theta - \mu g\cos\theta$ | if surface is rough but friction is insufficient to prevent slipping. (forward slipping will take place) |
| $a = \dfrac{g\sin\theta}{1 + I/MR^2}$ | if pure rolling is taking place, *i.e.*, friction is sufficient to prevent slipping. |

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# Solved Examples

## Level 1

**Example 1** *If the radius of the earth contracts to half of its present value without change in its mass, what will be the new duration of the day?*

**Solution** Present angular momentum of earth

$$L_1 = I\omega = \frac{2}{5}MR^2\omega$$

New angular momentum because of change in radius

$$L_2 = \frac{2}{5}M\left(\frac{R}{2}\right)^2 \omega'$$

If external torque is zero then angular momentum must be conserved

$$L_1 = L_2$$

$$\frac{2}{5}MR^2\omega = \frac{1}{4}\times\frac{2}{5}MR^2\omega'$$

*i.e.,*

$$\omega' = 4\omega$$

$$T' = \frac{1}{4}T = \frac{1}{4}\times 24 = 6\,\text{h}$$

**Example 2** *A particle of mass m is projected with velocity v at an angle θ with the horizontal. Find its angular momentum about the point of projection when it is at the highest point of its trajectory.*

**Solution** At the highest point it has only horizontal velocity $v_x = v\cos\theta$

Length of the perpendicular to the horizontal velocity from '$O$' is the maximum height, where

$$H_{\max} = \frac{v^2\sin^2\theta}{2g}$$

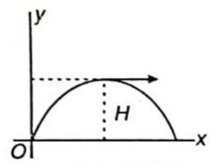


Fig. 9.86

$\Rightarrow$ Angular momentum $L = \dfrac{mv^3\sin^2\theta\cos\theta}{2g}$

**Example 3** *A horizontal force F acts on the sphere at its centre as shown. Coefficient of friction between ground and sphere is μ. What is maximum value of F, for which there is no slipping ?*

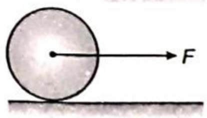


Fig. 9.87

**Solution**

$$F - f = Ma$$

$$f \cdot R = \frac{2}{5} MR^2 \frac{a}{R} \Rightarrow f = \frac{2}{5} Ma$$

$$\Rightarrow \qquad f = \frac{2}{7} F$$

$$\frac{2}{7} F \leq \mu mg \Rightarrow F \leq \frac{7}{2} \mu mg$$

**Example 4** *A tangential force F acts at the top of a thin spherical shell of mass m and radius R. Find the acceleration of the shell if it rolls without slipping.*

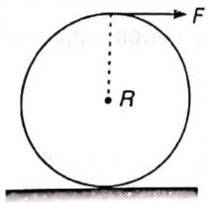


**Fig. 9.88**

**Solution** Let $f$ be the force of friction between the shell and the horizontal surface.

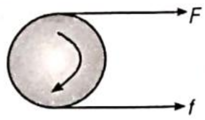


**Fig. 9.89**

For translational motion,

$$F + f = ma \qquad \ldots\text{(i)}$$

For rotational motion,

$$FR - fR = I\alpha = I\frac{a}{R} \qquad [\because a = R\alpha \text{ for pure rolling}]$$

$$\Rightarrow \qquad F - f = I\frac{a}{R^2} \qquad \ldots\text{(ii)}$$

Adding Eqs. (i) and (ii), we get

$$2F = \left(m + \frac{I}{R^2}\right)a = \left(m + \frac{2}{3}m\right)a = \frac{5}{3}ma$$

or

$$F = \frac{5}{6}ma \qquad \left[\because I_{\text{shell}} = \frac{2}{3}mR^2\right]$$

$$\Rightarrow \qquad a = \frac{6F}{5m}$$

**Example 5** *A solid cylinder of mass m and radius r starts rolling down an inclined plane of inclination θ. Friction is enough to prevent slipping. Find the speed of its centre of mass when its centre of mass has fallen a height h.*

**Solution** Considering the two shown positions of the cylinder. As it does not slip hence total mechanical energy will be conserved.

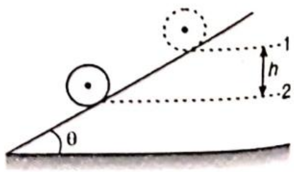


**Fig. 9.90**

Energy at postion 1 is $\qquad E_1 = mgh$

Energy at position 2 is $\qquad E_2 = \frac{1}{2} mv_{\text{COM}}^2 + \frac{1}{2} I_{\text{COM}} \omega^2$

$$\because \qquad \frac{v_{\text{COM}}}{r} = \omega \quad \text{and} \quad I_{\text{COM}} = \frac{mr^2}{2}$$

$\Rightarrow$ $$E_2 = \frac{3}{4} mv_{COM}^2$$

From COE, $E_1 = E_2$

$\Rightarrow$ $$v_{COM} = \sqrt{\frac{4}{3} gh}$$

**Example 6** *A disc starts rotating with constant angular acceleration of $\pi$ rad/s$^2$ about a fixed axis perpendicular to its plane and through its centre. Find :*

*(a) the angular velocity of the disc after 4 s.*

*(b) the angular displacement of the disc after 4 s and*

**Solution** Here $\alpha = \pi \,\text{rad/s}^2$, $\omega_0 = 0$, $t = 4\,\text{s}$

(a) $\omega_{(4\,s)} = 0 + (\pi\,\text{rad/s}^2) \times 4\,\text{s} = 4\pi\,\text{rad/s}.$

(b) $\theta_{(4\,s)} = 0 + \frac{1}{2}(\pi\,\text{rad/s}^2) \times (16\text{s}^2) = 8\pi\,\text{rad}$

**Example 7** *A small solid cylinder of radius r is released coaxially from point A inside the fixed large cylindrical bowl of radius R as shown in figure. If the friction between the small and the large cylinder is sufficient enough to prevent any slipping, then find :*

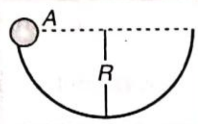



Fig. 9.91

*(a) What fractions of the total energy are translational and rotational, when the small cylinder reaches the bottom of the larger one?*

*(b) The normal force exerted by the small cylinder on the larger one when it is at the bottom.*

**Solution** (a) $$K_{trans} = \frac{1}{2} mv^2$$

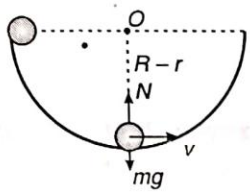



Fig. 9.92

$$K_{rot} = \frac{1}{2} I\omega^2 = \frac{1}{4} mv^2$$

$$K = K_{trans} + K_{rot} = \frac{3}{4} mv^2$$

$\therefore$ $$\frac{K_{trans}}{K} = \frac{2}{3}$$

$$\frac{K_{rot}}{K} = \frac{1}{3}$$

(b) From conservation of energy,

$$mg\,(R - r) = \frac{3}{4} mv^2$$

$\therefore$ $$\frac{mv^2}{R - r} = \frac{4}{3} mg$$

Now, $$N - mg = \frac{mv^2}{R - r}$$

$\therefore$ $$N = \frac{7}{3} mg$$

**Example 8** *A wheel rotates around a stationary axis so that the rotation angle* $\theta$ *varies with time as* $\theta = at^2$, *where* $a = 0.2\,rad/s^2$. *Find the magnitude of net acceleration of the point A at the rim at the moment* $t = 2.5\,s$ *if the linear velocity of the point A at this moment is* $v = 0.65\,m/s$.

**Solution** Instantaneous angular velocity at time $t$ is

$$\omega = \frac{d\theta}{dt} = \frac{d}{dt}(at^2)$$

or $$\omega = 2at = 0.4t \qquad (\text{as } a = 0.2 \text{ rad/s}^2)$$

Further, instantaneous angular acceleration is,

$$\alpha = \frac{d\omega}{dt} = \frac{d}{dt}(0.4t)$$

or $$\alpha = 0.4 \text{ rad/s}^2$$

Angular velocity at $t = 2.5\text{s}$ is

$$\omega = 0.4 \times 2.5 = 1.0 \text{ rad/s}$$

Further, radius of the wheel $R = \frac{v}{\omega}$ or $R = \frac{0.65}{1.0} = 0.65\text{ m}$

Now, magnitude of total acceleration is,

$$a = \sqrt{a_n^2 + a_t^2}$$

Here, $$a_n = R\omega^2 = (0.65)(1.0)^2 = 0.65 \text{ m/s}^2$$

and $$a_t = R\alpha = (0.65)(0.4) = 0.26 \text{ m/s}^2$$

$\therefore$ $$a = \sqrt{(0.65)^2 + (0.26)^2}$$

or $$a = 0.7 \text{ m/s}^2$$

**Example 9** *A solid ball of radius* 0.2 *m and mass* 1 *kg is given an instantaneous impulse of* 50 *N-s at point P as shown. Find the number of rotations made by the ball about its diameter before hitting the ground. The ball is kept on smooth surface initially.*

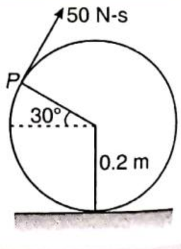



Fig. 9.93

**Solution** Impulse gives translational velocity

$$u = \frac{\text{Impulse}}{\text{Mass}} \text{ along impulse} = 50 \text{ m/s}$$

$T$ = time of flight of projectile

$$= \frac{2u\sin\theta}{g} = \frac{2 \times 50 \times \sin 60^\circ}{10} = 5\sqrt{3}\text{ sec}$$

Impulse give angular impulse also

$$\omega = \frac{\text{Impulse} \times R}{I}$$

or

$$\omega = \frac{\text{Impulse} \times R}{\frac{2}{5} mR^2}$$

Number of rotations,

$$n = \frac{\omega T}{2\pi} = \frac{3125\sqrt{3}}{2\pi}$$

## Level 2

**Example 1** *A solid ball rolls down a parabolic path ABC from a height h as shown in figure. Portion AB of the path is rough while BC is smooth. How high will the ball climb in BC?*

**Hint** *In case of pure rolling mechanical energy is conserved.*


**Fig. 9.94**

**Solution** At $B$, total kinetic energy $= mgh$

Here, $m =$ mass of ball

The ratio of rotational to translational kinetic energy would be,

$$\frac{K_R}{K_T} = \frac{2}{5}$$

$$\therefore \qquad K_R = \frac{2}{7} mgh \quad \text{and} \quad K_T = \frac{5}{7} mgh$$

In portion $BC$, friction is absent. Therefore, rotational kinetic energy will remain constant and translational kinetic energy will convert into potential energy. Hence, if $H$ be the height to which ball climbs in $BC$, then

$$mgH = K_T$$

or

$$mgH = \frac{5}{7} mgh \quad \text{or} \quad H = \frac{5}{7} h$$

**Example 2** *A thread is wound around two discs on either sides. The pulley and the two discs have the same mass and radius. There is no slipping at the pulley and no friction at the hinge. Find out the accelerations of the two discs and the angular acceleration of the pulley.*

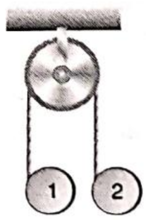



**Fig. 9.95**

**Solution** Let $R$ be the radius of the discs and $T_1$ and $T_2$ be the tensions in the left and right segments of the rope.

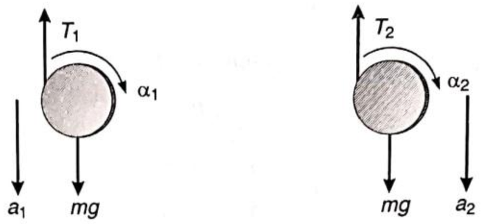



**Fig. 9.96**

Acceleration of disc 1,

$$a_1 = \frac{mg - T_1}{m} \quad \ldots(i)$$

Acceleration of disc 2,

$$a_2 = \frac{mg - T_2}{m} \quad \ldots(ii)$$

Angular acceleration of disc 1, $$\alpha_1 = \frac{\tau}{I} = \frac{T_1 R}{\frac{1}{2} mR^2} = \frac{2T_1}{mR} \quad \ldots(iii)$$

Similarly, angular acceleration of disc 2, $$\alpha_2 = \frac{2T_2}{mR} \quad \ldots(iv)$$

Both $\alpha_1$ and $\alpha_2$ are clockwise.

Angular acceleration of pulley,

$$\alpha = \frac{(T_2 - T_1)R}{\frac{1}{2} mR^2} = \frac{2(T_2 - T_1)}{mR} \quad \ldots(v)$$

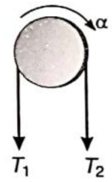



**Fig. 9.97**

For no slipping, $$R\alpha_1 - a_1 = a_2 - R\alpha_2 = R\alpha \quad \ldots(vi)$$

Solving these equations, we get

$$\alpha = 0 \quad \text{and} \quad a_1 = a_2 = \frac{2g}{3}$$

**Alternate Solution**

As both the discs are in identical situation, $T_1 = T_2$ and $\alpha = 0$. *i.e.*, each of the discs falls independently and identically. Therefore, this is exactly similar to the problem shown in figure.

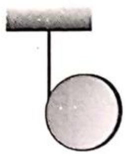


**Fig. 9.98**

**Example 3** *A thin massless thread is wound on a reel of mass* 3 kg *and moment of inertia* 0.6 kg-m$^2$. *The hub radius is* $R = 10$ cm *and peripheral radius is* $2R = 20$ cm *The reel is placed on a rough table and the friction is enough to prevent slipping. Find the acceleration of the centre of reel and of hanging mass of* 1 kg.

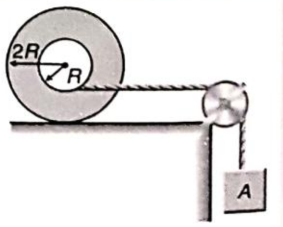



**Fig. 9.99**

**Solution** Let, $a_1$ = acceleration of centre of mass of reel

$a_2$ = acceleration of 1 kg block

$\alpha$ = angular acceleration of reel (clockwise)

$T$ = tension in the string

and $f$ = force of friction

Free body diagram of reel is as shown below: (only horizontal forces are shown).

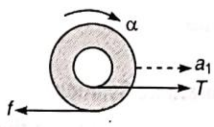


Fig. 9.100

Equations of motion are :

$$T - f = 3a_1 \qquad \text{...(i)}$$

$$\alpha = \frac{\tau}{I} = \frac{f(2R) - T.R}{I} = \frac{0.2f - 0.1T}{0.6} = \frac{f}{3} - \frac{T}{6} \qquad \text{...(ii)}$$

Free body diagram of mass is,

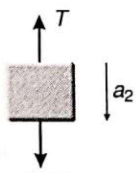


Fig. 9.101

Equation of motion is,

$$10 - T = a_2 \qquad \text{...(iii)}$$

For no slipping condition,

$$a_1 = 2R\alpha \quad \text{or} \quad a_1 = 0.2\alpha \qquad \text{...(iv)}$$

and

$$a_2 = a_1 - R\alpha \quad \text{or} \quad a_2 = a_1 - 0.1\alpha \qquad \text{...(v)}$$

Solving the above five equations, we get

$$a_1 = 0.27 \text{ m/s}^2$$

and

$$a_2 = 0.135 \text{ m/s}^2$$

**Example 4** *A solid sphere of radius r is gently placed on a rough horizontal ground with an initial angular speed $\omega_0$ and no linear velocity. If the coefficient of friction is $\mu$, find the time t when the slipping stops. In addition, state the linear velocity v and angular velocity $\omega$ at the end of slipping.*

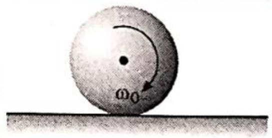


Fig. 9.102

**Solution** Let $m$ be the mass of the sphere.

Since, it is a case of backward slipping, force of friction is in forward direction. Limiting friction will act in this case.

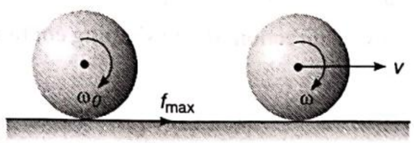


Fig. 9.103

Linear acceleration
$$a = \frac{f}{m} = \frac{\mu mg}{m} = \mu g$$

Angular retardation
$$\alpha = \frac{\tau}{I} = \frac{f \cdot r}{\frac{2}{5}mr^2} = \frac{5}{2}\frac{\mu g}{r}$$

Slipping is ceased when.
$$v = r\omega$$

or
$$(at) = r(\omega_0 - \alpha t)$$

or
$$\mu g t = r\left(\omega_0 - \frac{5}{2}\frac{\mu g t}{r}\right)$$

or $$\frac{7}{2}\mu g t = r\omega_0$$

$\therefore$ $$t = \frac{2}{7}\frac{r\omega_0}{\mu g}$$

$$v = at = \mu g t = \frac{2}{7} r\omega_0$$

and $$\omega = \frac{v}{r} = \frac{2}{7}\omega_0$$

**Alternate Solution**

Net torque on the sphere about the bottommost point is zero. Therefore, angular momentum of the sphere will remain conserved about the bottommost point.

$$L_i = L_f$$

$\therefore$ $$I\omega_0 = I\omega + mrv$$

or $$\frac{2}{5}mr^2\omega_0 = \frac{2}{5}mr^2\omega + mr(\omega r)$$

$\therefore$ $$\omega = \frac{2}{7}\omega_0 \quad \text{and} \quad v = r\omega = \frac{2}{7}r\omega_0$$

**Example 5** *A billiard ball, initially at rest, is given a sharp impulse by a cue. The cue is held horizontally a distance h above the centre line as shown in figure. The ball leaves the cue with a speed $v_0$ and because of its forward english (backward slipping) eventually acquires a final speed $\frac{9}{7}v_0$. Show that*

$$h = \frac{4}{5}R$$

*where R is the radius of the ball.*

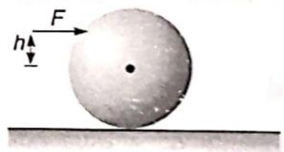



Fig. 9.104

**Solution** Let $\omega_0$ be the angular speed of the ball just after it leaves the cue. The maximum friction acts in forward direction till the slipping continues. Let $v$ be the linear speed and $\omega$ the angular speed when slipping is ceased.

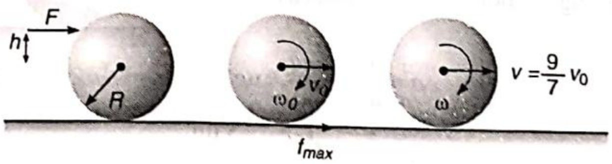



Fig. 9.105

$\therefore$ $$v = R\omega \quad \text{or} \quad \omega = \frac{v}{R}$$

Given, $$v = \frac{9}{7}v_0 \qquad \text{...(i)}$$

$\therefore$ $$\omega = \frac{9}{7}\frac{v_0}{R} \qquad \text{...(ii)}$$

Applying, Linear impulse = change in linear momentum

$\therefore$ $$F\,dt = mv_0 \qquad \ldots\text{(iii)}$$

Angular impulse = change in angular momentum

$\therefore$ $$\tau\,dt = I\omega_0$$

or $$Fh\,dt = \frac{2}{5}mR^2\,\omega_0 \qquad \ldots\text{(iv)}$$

Angular momentum about bottommost point will remain conserved.

*i.e.*, $$L_i = L_f$$

or $$I\omega_0 + mRv_0 = I\omega + mRv$$

$\therefore$ $$\frac{2}{5}mR^2\,\omega_0 + mRv_0 = \frac{2}{5}mR^2\left(\frac{9}{7}\frac{v_0}{R}\right) + \frac{9}{7}mRv_0 \qquad \ldots\text{(v)}$$

Solving Eqs. (iii), (iv) and (v), we get

$$h = \frac{4}{5}R$$ **Proved.**

**Example 6** *Determine the maximum horizontal force F that may be applied to the plank of mass m for which the solid sphere does not slip as it begins to roll on the plank. The sphere has a mass M and radius R. The coefficient of static and kinetic friction between the sphere and the plank are $\mu_s$ and $\mu_k$ respectively.*

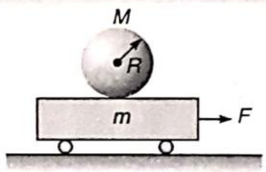



Fig. 9.106

**Solution** The free body diagrams of the sphere and the plank are as shown below:

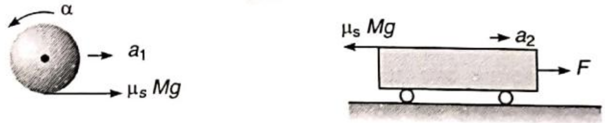



Fig. 9.107

Writing equations of motion :

For sphere : Linear acceleration $$a_1 = \frac{\mu_s Mg}{M} = \mu_s g \qquad \ldots\text{(i)}$$

Angular acceleration $$\alpha = \frac{(\mu_s Mg)R}{\frac{2}{5}MR^2} = \frac{5}{2}\frac{\mu_s g}{R} \qquad \ldots\text{(ii)}$$

For plank : Linear acceleration $$a_2 = \frac{F - \mu_s Mg}{m} \qquad \ldots\text{(iii)}$$

For no slipping : $$a_2 = a_1 + R\alpha \qquad \ldots\text{(iv)}$$

Solving the above four equations, we get $F = \mu_s g\left(M + \frac{7}{2}m\right)$

Thus, maximum value of $F$ can be $\mu_s g\left(M + \frac{7}{2}m\right)$

**Example 7** *A uniform disc of radius $r_0$ lies on a smooth horizontal plane. A similar disc spinning with the angular velocity $\omega_0$ is carefully lowered onto the first disc. How soon do both discs spin with the same angular-velocity if the friction coefficient between them is equal to $\mu$ ?*

**Solution** From the law of conservation of angular momentum.

$$I\omega_0 = 2I\omega$$

Here, $I$ = moment of inertia of each disc relative to common rotation axis

$\therefore$
$$\omega = \frac{\omega_0}{2} = \text{steady state angular velocity}$$

The angular velocity of each disc varies due to the torque $\tau$ of the friction forces. To calculate $\tau$, let us take an elementary ring with radii $r$ and $r + dr$. The torque of the friction forces acting on the given ring is equal to.

$$d\tau = \mu r\left(\frac{mg}{\pi r_0^2}\right)2\pi r\, dr = \left(\frac{2\mu mg}{r_0^2}\right)r^2\, dr$$

where $m$ is the mass of each disc. Integrating this with respect to $r$ between 0 and $r_0$, we get

$$\tau = \frac{2}{3}\mu mgr_0$$

The angular velocity of the lower disc increases by $d\omega$ over the time interval

$$dt = \left(\frac{I}{\tau}\right)d\omega = \left(\frac{3r_0}{4\mu g}\right)d\omega$$

Integrating this equation with respect to $\omega$ between 0 and $\frac{\omega_0}{2}$, we find the desired time

$$t = \frac{3r_0\,\omega_0}{8\mu g}$$

# EXERCISES

## AIEEE Corner

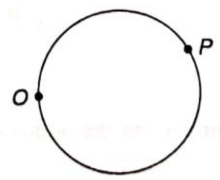
### Subjective Questions (Level 1)

#### Moment of Inertia

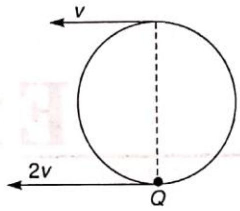
1. Four thin rods each of mass $m$ and length $l$ are joined to make a square. Find moment of inertia of all the four rods about any side of the square.
2. A mass of 1 kg is placed at (1 m, 2 m, 0). Another mass of 2 kg is placed at (3 m, 4 m, 0). Find moment of inertia of both the masses about $z$-axis.
3. Moment of inertia of a uniform rod of mass $m$ and length $l$ is $\frac{7}{12}ml^2$ about a line perpendicular to the rod. Find the distance of this line from the middle point of the rod.
4. Find the moment of inertia of a uniform square plate of mass $M$ and edge $a$ about one of its diagonals.
5. Radius of gyration of a body about an axis at a distance 6 cm from its centre of mass is 10 cm. Find its radius of gyration about a parallel axis through its centre of mass.
6. Two point masses $m_1$ and $m_2$ are joined by a weightless rod of length $r$. Calculate the moment of inertia of the system about an axis passing through its centre of mass and perpendicular to the rod.
7. Linear mass density (mass/length) of a rod depends on the distance from one end (say $A$) as $\lambda_x = (\alpha x + \beta)$. Here, $\alpha$ and $\beta$ are constants. Find the moment of inertia of this rod about an axis passing through $A$ and perpendicular to the rod. Length of the rod is $l$.

#### Angular Velocity

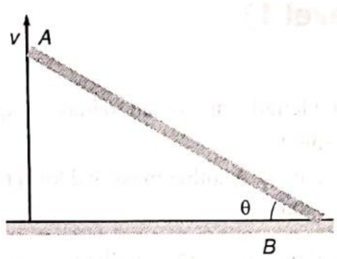
8. Find angular speed of second's clock.
9. A particle is located at (3 m, 4 m) and moving with $\vec{v} = (4\hat{i} - 3\hat{j})$ m/s. Find its angular velocity about origin at this instant.
10. Particle $P$ shown in figure is moving in a circle of radius $R = 10$ cm with linear speed $v = 2$ m/s. Find the angular speed of particle about point $O$.



**11.** Two points $P$ and $Q$, diametrically opposite on a disc of radius $R$ have linear velocities $v$ and $2v$ as shown in figure. Find the angular speed of the disc.




**12.** Point $A$ of rod $AB$ ($l = 2\,\text{m}$) is moved upwards against a wall with velocity $v = 2\,\text{m/s}$. Find angular speed of the rod at an instant when $\theta = 60°$.

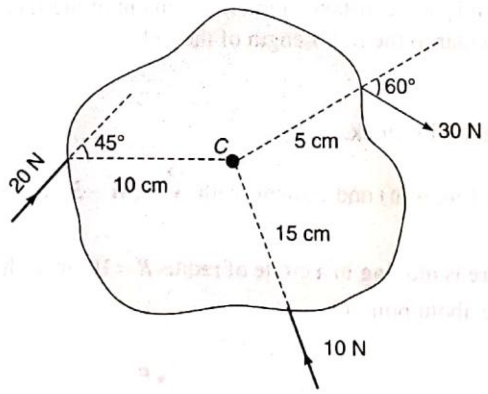



## Torque

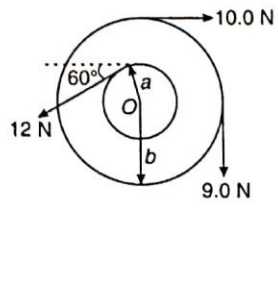
**13.** A force $\vec{\mathbf{F}} = (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 2\hat{\mathbf{k}})$ N is acting on a body at point (2 m, 4 m, –2 m). Find torque of this force about origin.

**14.** A particle of mass $m = 1\,\text{kg}$ is projected with speed $u = 20\sqrt{2}\,\text{m/s}$ at angle $\theta = 45°$ with horizontal. Find the torque of the weight of the particle about the point of projection when the particle is at the highest point.

**15.** Point $C$ is the centre of mass of the rigid body shown in figure. Find the total torque acting on the body about point $C$.




**16.** Find the net torque on the wheel in figure about the point $O$ if $a = 10\,\text{cm}$ and $b = 25\,\text{cm}$.



## Rotation of a Rigid Body About a Fixed Axis

### Uniform angular acceleration

**17.** A wheel rotating with uniform angular acceleration covers 50 rev in the first five seconds after the start. Find the angular acceleration and the angular velocity at the end of five seconds.

**18.** A wheel starting from rest is uniformly accelerated with $\alpha = 2\ \text{rad/s}^2$ for 5 s. It is then allowed to rotate uniformly for the next two seconds and is finally brought to rest in the next 5 s. Find the total angle rotated by the wheel.

**19.** A wheel whose moment of inertia is $0.03\ \text{kg m}^2$, is accelerated from rest to 20 rad/s in 5 s. When the external torque is removed, the wheel stops in 1 min. Find :

(a) the frictional torque, (b) the external torque.

**20.** A body rotating at 20 rad/s is acted upon by a constant torque providing it a deceleration of $2\ \text{rad/s}^2$. At what time will the body have kinetic energy same as the initial value if the torque continues to act ?

**21.** A uniform disc of mass 20 kg and radius 0.5 m can turn about a smooth axis through its centre and perpendicular to the disc. A constant torque is applied to the disc for 3 s from rest and the angular velocity at the end of that time is $\frac{240}{\pi}$ rev/min. Find the magnitude of the torque. If the torque is then removed and the disc is brought to rest in $t$ seconds by a constant force of 10 N applied tangentially at a point on the rim of the disc, find $t$.

**22.** A uniform disc of mass $m$ and radius $R$ is rotated about an axis passing through its centre and perpendicular to its plane with an angular velocity $\omega_0$. It is placed on a rough horizontal plane with the axis of the disc keeping vertical. Coefficient of friction between the disc and the surface is $\mu$. Find :

(a) the time when disc stops rotating,

(b) the angle rotated by the disc before stopping.

### Non-uniform angular acceleration

**23.** A flywheel whose moment of inertia about its axis of rotation is 16 kg-m$^2$ is rotating freely in its own plane about a smooth axis through its centre. Its angular velocity is $9\ \text{rad s}^{-1}$ when a torque is applied to bring it to rest in $t_0$ seconds. Find $t_0$ if :

(a) the torque is constant and of magnitude of 4 Nm,

(b) the magnitude of the torque after $t$ seconds is given by $kt$.

**24.** A shaft is turning at 65 rad/s at time zero. Thereafter, angular acceleration is given by $\alpha = -10\ \text{rad/s}^2 - 5t\ \text{rad/s}^2$

where $t$ is the elapsed time.

(a) Find its angular speed at $t = 3.0$ s.

(b) How far does it turn in these 3s ?

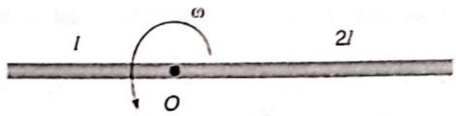
**25.** The angular velocity of a gear is controlled according to $\omega = 12 - 3t^2$ where $\omega$, in radian per second, is positive in the clockwise sense and $t$ is the time in seconds. Find the net angular displacement $\Delta\theta$ from the time $t = 0$ to $t = 3$ s. Also, find the number of revolutions $N$ through which the gear turns during the 3 s.

**26.** A solid body rotates about a stationary axis according to the law $\theta = at - bt^3$, where $a = 6$ rad/s and $b = 2\ \text{rad/s}^3$. Find the mean values of the angular velocity and acceleration over the time interval between $t = 0$ and the time, when the body comes to rest.

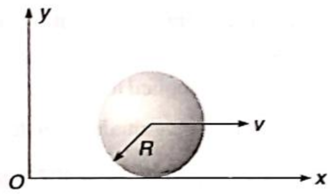
## Angular Momentum

**27.** A particle of mass 1 kg is moving along a straight line $y = x + 4$. Both $x$ and $y$ are in metres. Velocity of the particle is 2 m/s. Find magnitude of angular momentum of the particle about origin.

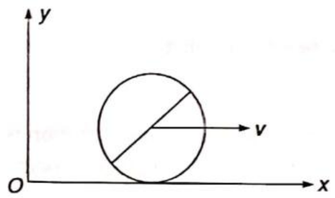
**28.** A uniform rod of mass $m$ is rotated about an axis passing through point $O$ as shown. Find angular momentum of the rod about rotational axis.




**29.** A solid sphere of mass $m$ and radius $R$ is rolling without slipping as shown in figure. Find angular momentum of the sphere about $z$-axis.



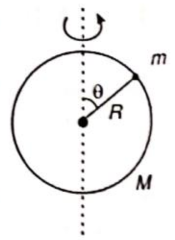

**30.** A rod of mass $m$ and length $2R$ is fixed along the diameter of a ring of same mass $m$ and radius $R$ as shown in figure. The combined body is rolling without slipping along $x$-axis. Find the angular momentum about $z$-axis.




## Conservation of Angular Momentum

**31.** If radius of earth is increased, without change in its mass, will the length of day increase, decrease or remain same?

**32.** The figure shows a thin ring of mass $M = 1$ kg and radius $R = 0.4$ m spinning about a vertical diameter. (Take $I = \frac{1}{2}MR^2$). A small bead of mass $m = 0.2$ kg can slide without friction along the ring. When the bead is at the top of the ring, the angular velocity is 5 rad/s. What is the angular velocity when the bead slips halfway to $\theta = 45°$.




**33.** A horizontal disc rotating freely about a vertical axis makes 100 rpm. A small piece of wax of mass 10 g falls vertically on the disc and adheres to it at a distance of 9 cm from the axis. If the number of revolutions per minute is thereby reduced to 90. Calculate the moment of inertia of disc.

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**34.** A man stands at the centre of a circular platform holding his arms extended horizontally with 4 kg block in each hand. He is set rotating about a vertical axis at 0.5 rev/s. The moment of inertia of the man plus platform is 1.6 kg-m$^2$, assumed constant. The blocks are 90 cm from the axis of rotation. He now pulls the blocks in toward his body until they are 15 cm from the axis of rotation. Find (a) his new angular velocity and (b) the initial and final kinetic energy of the man and platform. (c) how much work must the man do to pull in the blocks ?

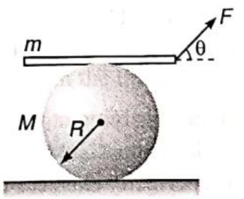
**35.** A horizontally oriented uniform disc of mass $M$ and radius $R$ rotates freely about a stationary vertical axis passing through its centre. The disc has a radial guide along which can slide without friction a small body of mass $m$. A light thread running down through the hollow axle of the disc is tied to the body. Initially the body was located at the edge of the disc and the whole system rotated with an angular velocity $\omega_0$. Then, by means of a force $F$ applied to the lower end of the thread the body was slowly pulled to the rotation axis. Find :

(a) the angular velocity of the system in its final state,

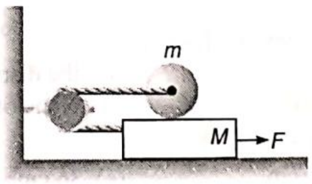
(b) the work performed by the force $F$.

## Pure Rolling

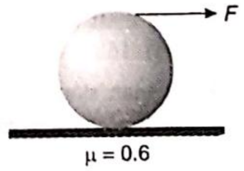
**36.** Consider a cylinder of mass $M$ and radius $R$ lying on a rough horizontal plane. It has a plank lying on its top as shown in figure. A force $F$ is applied on the plank such that the plank moves and causes the cylinder to roll. The plank always remains horizontal. There is no slipping at any point of contact. Calculate the acceleration of the cylinder and the frictional forces at the two contacts.



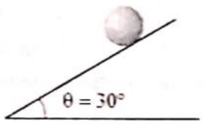

**37.** Find the acceleration of the cylinder of mass $m$ and radius $R$ and that of plank of mass $M$ placed on smooth surface if pulled with a force $F$ as shown in figure. Given that sufficient friction is present between cylinder and the plank surface to prevent sliding of cylinder.




**38.** In the figure shown a force $F$ is applied at the top of a disc of mass 4 kg and radius 0.25 m. Find maximum value of $F$ for no slipping



**39.** In the figure shown a solid sphere of mass 4 kg and radius 0.25 m is placed on a rough surface. Find : $(g = 10 \text{ m/s}^2)$




(a) minimum coefficient of friction for pure rolling to take place.

(b) If $\mu > \mu_{min}$, find linear acceleration of sphere.

(c) If $\mu = \frac{\mu_{min}}{2}$, find linear acceleration of cylinder.

Here, $\mu_{min}$ is the value obtained in part (a).

## Angular Impulse

**40.** A uniform rod $AB$ of length $2l$ and mass $m$ is rotating in a horizontal plane about a vertical axis through $A$, with angular velocity $\omega$ when the mid-point of the rod strikes a fixed nail and is brought immediately to rest. Find the impulse exerted by the nail.

**41.** A uniform rod of length $L$ rests on a frictionless horizontal surface. The rod is pivoted about a fixed frictionless axis at one end. The rod is initially at rest. A bullet travelling parallel to the horizontal surface and perpendicular to the rod with speed $v$ strikes the rod at its centre and becomes embedded in it. The mass of the bullet is one-sixth the mass of the rod.

(a) What is the final angular velocity of the rod ?

(b) What is the ratio of the kinetic energy of the system after the collision to the kinetic energy of the bullet before the collision ?

**42.** A uniform rod $AB$ of mass $3m$ and length $2l$ is lying at rest on a smooth horizontal table with a smooth vertical axis through the end $A$. A particle of mass $2m$ moves with speed $2u$ across the table and strikes the rod at its mid-point $C$. If the impact is perfectly elastic. Find the speed of the particle after impact if :

(a) it strikes the rod normally,

(b) its path before impact was inclined at 60° to $AC$.

# Objective Questions (Level 1)

## Single Correct Option

**1.** The moment of inertia of a body does not depend on

(a) mass of the body
(b) the distribution of the mass in the body
(c) the axis of rotation of the body
(d) None of these

**2.** The radius of gyration of a disc of radius 25 cm is

(a) 18 cm (b) 12.5 cm (c) 36 cm (d) 50 cm

**3.** A shaft initially rotating at 1725 rpm is brought to rest uniformly in 20s. The number of revolutions that the shaft will make during this time is

(a) 1680 (b) 575 (c) 287 (d) 627

**4.** A man standing on a platform holds weights in his outstretched arms. The system is rotated about a central vertical axis. If the man now pulls the weights inwards close to his body, then

(a) the angular velocity of the system will increase

(b) the angular momentum of the system will remain constant

(c) the kinetic energy of the system will increase
(d) All of the above

**5.** The moment of inertia of a uniform semicircular disc of mass $M$ and radius $r$ about a line perpendicular to the plane of the disc through the centre is

(a) $Mr^2$ (b) $\frac{1}{2}Mr^2$ (c) $\frac{1}{4}Mr^2$ (d) $\frac{2}{5}Mr^2$

**6.** Two bodies $A$ and $B$ made of same material have the moment of inertial in the ratio $I_A : I_B = 16:18$. The ratio of the masses $m_A : m_B$ is given by

(a) cannot be obtained (b) 2 : 3
(c) 1 : 1 (d) 4 : 9

**7.** When a sphere rolls down an inclined plane, then identity the correct statement related to the work done by friction force

(a) The friction force does positive translational work
(b) The friction force does negative rotational work
(c) The net work done by friction is zero
(d) All of the above

**8.** A circular table rotates about a vertical axis with a constant angular speed ω. A circular pan rests on the turn table (with the centre coinciding with centre of table) and rotates with the table. The bottom of the pan is covered with a uniform thick layer of ice which also rotates with the pan. The ice starts melting. The angular speed of the turn table

(a) remains the same
(b) decreases
(c) increases
(d) may increase or decrease depending on the thickness of ice layer

**9.** If $R$ is the radius of gyration of a body of mass $M$ and radius $r$, then the ratio of its rotational to translational kinetic energy in the rolling condition is

(a) $\frac{R^2}{R^2 + r^2}$ (b) $\frac{R^2}{r^2}$ (c) $\frac{r^2}{R^2}$ (d) 1

**10.** A solid sphere rolls down two different inclined planes of the same height but of different inclinations

(a) in both cases the speeds and time of descend will be same
(b) the speeds will be same but time of descend will be different
(c) the speeds will be different but time of descend will be same
(d) speeds and time of descend both will be different

**11.** For the same total mass which of the following will have the largest moment of inertia about an axis passing through the centre of mass and perpendicular to the plane of the body

(a) a disc of radius $R$ (b) a ring of radius $R$
(c) a square lamina of side $2R$ (d) four rods forming a square of side $2R$

**12.** A disc and a solid sphere of same mass and radius roll down an inclined plane. The ratio of the friction force acting on the disc and sphere is

(a) $\frac{7}{6}$ (b) $\frac{5}{4}$
(c) $\frac{3}{2}$ (d) depends on angle of inclination

**13.** A horizontal disc rotates freely with angular velocity $\omega$ about a vertical axes through its centre. A ring, having the same mass and radius as the disc, is now gently placed coaxially on the disc. After some time, the two rotate with a common angular velocity. Then

(a) no friction exists between the disc and the ring
(b) the angular momentum of the system is conserved
(c) the final common angular velocity is $\frac{1}{2}\omega$
(d) All of the above

**14.** A solid homogeneous sphere is moving on a rough horizontal surface, partly rolling and partly sliding. During this kind of motion of the sphere

(a) total kinetic energy of the sphere is conserved
(b) angular momentum of the sphere about any point on the horizontal surface is conserved
(c) only the rotational kinetic energy about the centre of mass is conserved
(d) None of the above

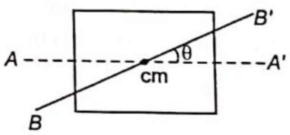
**15.** A particle of mass $m = 3$ kg moves along a straight line $4y - 3x = 2$ where $x$ and $y$ are in metre, with constant velocity $v = 5\ \text{ms}^{-1}$. The magnitude of angular momentum about the origin is

(a) $12\ \text{kg m}^2\text{s}^{-1}$ (b) $6.0\ \text{kg m}^2\text{s}^{-1}$ (c) $4.5\ \text{kg m}^2\text{s}^{-1}$ (d) $8.0\ \text{kg m}^2\text{s}^{-1}$

**16.** A solid sphere rolls without slipping on a rough horizontal floor, moving with a speed $v$. It makes an elastic collision with a smooth vertical wall. After impact,

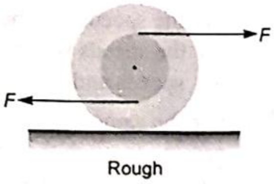
(a) it will move with a speed $v$ initially
(b) its motion will be rolling with slipping initially and its rotational motion will stop momentarily at some instant
(c) its motion will be rolling without slipping only after some time
(d) All of the above

**17.** The figure shows a square plate of uniform mass distribution. $AA'$ and $BB'$ are the two axes lying in the plane of the plate and passing through its centre of mass. It $I_o$ is the moment of inertia of the plate about $AA'$ then its moment of inertia about the $BB'$ axis is




(a) $I_o$ (b) $I_o \cos\theta$
(c) $I_o \cos^2\theta$ (d) None of these

**18.** A spool is pulled horizontally on rough surface by two equal and opposite forces as shown in the figure. Which of the following statements are correct?




(a) The centre of mass moves towards left
(b) The centre of mass moves towards right
(c) The centre of mass remains stationary
(d) The net torque about the centre of mass of the spool is zero

**19.** Two identical discs are positioned on a vertical axis as shown in the figure. The bottom disc is rotating at angular velocity $\omega_0$ and has rotational kinetic energy $K_0$. The top disc is initially at rest. It then falls and sticks to the bottom disc. The change in the rotational kinetic energy of the system is

(a) $K_0/2$ (b) $-K_0/2$
(c) $-K_0/4$ (d) $K_0/4$

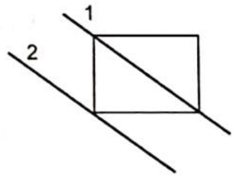
**20.** The moment of inertia of hollow sphere (mass $M$) of inner radius $R$ and outer radius $2R$, having material of uniform density, about a diametric axis is

(a) $31\,MR^2/70$ (b) $43\,MR^2/90$ (c) $19\,MR^2/80$ (d) None of these

**21.** A rod of uniform cross-section of mass $M$ and length $L$ is hinged about an end to swing freely in a vertical plane. However, its density is non uniform and varies linearly from hinged end to the free end doubling its value. The moment of inertia of the rod, about the rotation axis passing through the hinge point is

(a) $\frac{2ML^2}{9}$ (b) $\frac{3ML^2}{16}$ (c) $\frac{7ML^2}{18}$ (d) None of these

**22.** Let $I_1$ and $I_2$ be the moment of inertia of a uniform square plate about axes shown in the figure. Then the ratio $I_1 : I_2$ is



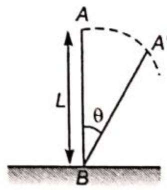

(a) $1:\frac{1}{7}$ (b) $1:\frac{12}{7}$

(c) $1:\frac{7}{12}$ (d) $1:7$

**23.** Moment of inertia of a uniform rod of length $L$ and mass $M$, about an axis passing through $L/4$ from one end and perpendicular to its length is

(a) $\frac{7}{36}ML^2$ (b) $\frac{7}{48}ML^2$ (c) $\frac{11}{48}ML^2$ (d) $\frac{ML^2}{12}$

**24.** A uniform rod of length $L$ is free to rorate in a vertical plane about a fixed horizontal axis through $B$. The rod begins rotating from rest. The angular velocity $\omega$ at angle $\theta$ is given as



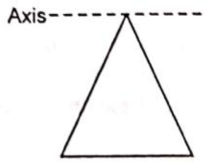

(a) $\sqrt{\left(\frac{6g}{L}\right)}\sin\frac{\theta}{2}$ (b) $\sqrt{\left(\frac{6g}{L}\right)}\cos\frac{\theta}{2}$

(c) $\sqrt{\left(\frac{6g}{L}\right)}\sin\theta$ (d) $\sqrt{\left(\frac{6g}{L}\right)}\cos\theta$

**25.** Two particles of masses $m_1$ and $m_2$ are connected by a light rod of length $r$ to constitute a dumb-bell. The moment of inertia of the dumb-bell about an axis perpendicular to the rod passing through the centre of mass of the two particles is

(a) $\frac{m_1 m_2 r^2}{m_1 + m_2}$ (b) $(m_1 + m_2)r^2$ (c) $\frac{m_1 m_2 r^2}{m_1 - m_2}$ (d) $(m_1 - m_2)r^2$

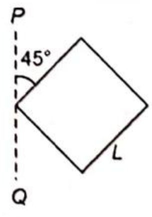
**26.** Find moment of inertia of a thin sheet of mass $M$ in the shape of an equilateral triangle about an axis as shown in figure. The length of each side is $L$




(a) $ML^2/8$ (b) $3ML^2/8$

(c) $7ML^2/8$ (d) None of these

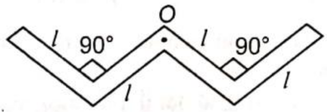
**27.** A square is made by joining four rods each of mass $M$ and length $L$. Its moment of inertia about an axis $PQ$, in its plane and passing through one of its corner is




(a) $6\,ML^2$ (b) $\frac{4}{3}ML^2$

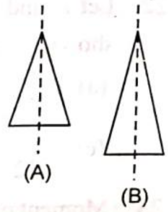
(c) $\frac{8}{3}ML^2$ (d) $\frac{10}{3}ML^2$

**28.** A thin rod of length $4l$, mass $4m$ is bent at the points as shown in the figure. What is the moment of inertia of the rod about the axis passing through $O$ and perpendicular to the plane of the paper?



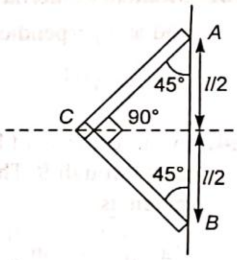

(a) $\frac{ml^2}{3}$ (b) $\frac{10ml^2}{3}$ (c) $\frac{ml^2}{12}$ (d) $\frac{ml^2}{24}$

**29.** The figure shows two cones $A$ and $B$ with the conditions : $h_A < h_B$; $\rho_A > \rho_B$; $R_A = R_B$ $m_A = m_B$. Identify the correct statement about their axis of symmetry.




(a) Both have same moment of inertia
(b) A has greater moment of inertia
(c) B has greater moment of inertia
(d) Nothing can be said

**30.** Linear mass density of the two rods system, $AC$ and $CB$ is $x$. Moment of inertia of two rods about an axis passing through $AB$ is




(a) $\frac{xl^3}{4\sqrt{3}}$ (b) $\frac{xl^3}{\sqrt{2}}$

(c) $\frac{xl^3}{4}$ (d) $\frac{xl^3}{6\sqrt{2}}$

# JEE Corner

## Assertion and Reason

**Directions :** *Choose the correct option.*

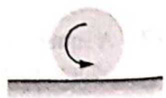
(a) *If both* ***Assertion*** *and* ***Reason*** *are true and the* ***Reason*** *is correct explanation of the* ***Assertion.***
(b) *If both* ***Assertion*** *and* ***Reason*** *are true but* ***Reason*** *is not the correct explanation of* ***Assertion.***
(c) *If* ***Assertion*** *is true, but the* ***Reason*** *is false.*
(d) *If* ***Assertion*** *is false but the* ***Reason*** *is true.*

**1. Assertion :** Moment of inertia of a rigid body about any axis passing through its centre of mass is minimum.

**Reason :** From theorem of parallel axis,

$$I = I_{cm} + Mr^2$$

**2. Assertion :** A ball is released on a rough ground in the condition shown in figure. It will start pure rolling after some time towards left side.

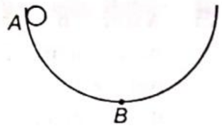


**Reason :** Friction will convert the pure rotational motion of the ball into pure rolling.

**3. Assertion :** A solid sphere and a hollow sphere are rolling on ground with same total kinetic energies. If translational kinetic energy of solid sphere is $K$, then translational kinetic energy of hollow sphere should be greater than $K$.

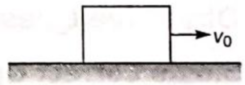
**Reason :** In case of hollow sphere rotational kinetic energy is less than its translational kinetic energy.

4. **Assertion :** A small ball is released from rest from point $A$ as shown. If bowl is smooth than ball will exert more pressure at point $B$, compared to the situation if bowl is rough.



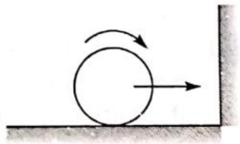

**Reason :** Linear velocity and hence, centripetal force in smooth situation is more.

5. **Assertion :** A cubical block is moving on a rough ground with velocity $v_0$. During motion net normal reaction on the block from ground will not pass through centre of cube. It will shift towards right.



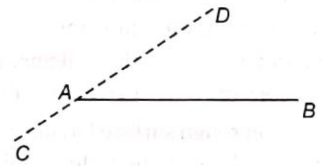

**Reason :** It is to keep the block in rotational equilibrium.

6. **Assertion :** A ring is rolling without slipping on a rough ground. It strikes elastically with a smooth wall as shown in figure. Ring will stop after some time while travelling in opposite direction.



**Reason :** Net angular momentum about an axis passing through bottommost point and perpendicular to plane of paper is zero.

7. **Assertion :** There is a thin rod $AB$ and a dotted line $CD$. All the axes we are talking about are perpendicular to plane of paper. As we take different axes moving from $A$ to $D$, moment of inertia of the rod may first decrease then increase.



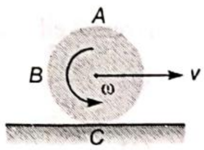

**Reason :** Theorem of perpendicular axis cannot be applied here.

8. **Assertion :** If linear momentum of a particle is constant, then its angular momentum about any axis will also remain constant.

**Reason :** Linear momentum remain constant, if $\vec{\mathbf{F}}_{\text{net}} = 0$ and angular momentum remains constant if $\vec{\tau}_{\text{net}} = 0$.

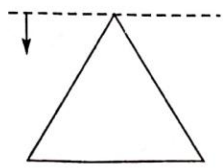
9. **Assertion :** In the figure shown, $A$, $B$ and $C$ are three points on the circumference of a disc. Let $v_A$, $v_B$ and $v_C$ are speeds of these three points, then

$$v_C > v_B > v_A$$



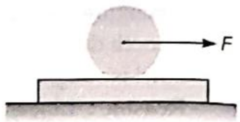

**Reason :** In case of rotational plus translational motion of a rigid body, net speed of any point (other than centre of mass) is greater than, less than or equal to the speed of centre of mass.

10. **Assertion :** There is a triangular plate as shown. A dotted axis is lying in the plane of slab. As the axis is moved downwards, moment of inertia of slab will first decrease then increase.



**Reason :** Axis is first moving towards its centre of mass and then it is receding from it.

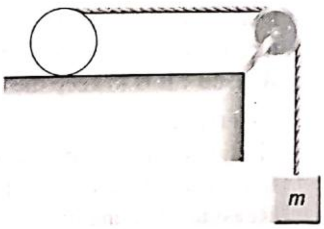
**11.** **Assertion :** A horizontal force $F$ is applied at the centre of solid sphere placed over a plank. The minimum coefficient of friction between plank and sphere required for pure rolling is $\mu_1$ when plank is kept at rest and $\mu_2$ when plank can move, then $\mu_2 < \mu_1$.




**Reason :** Work done by frictional force on the sphere in both cases is zero.

## Objective Questions (Level 2)

### Single Correct Option




**1.** In the given figure a ring of mass $m$ is kept on a horizontal surface while a body of equal mass $m$ is attached through a string, which is wounded on the ring. When the system is released, the ring rolls without slipping. Consider the following statement and choose the correct option.

(i) acceleration of the centre of mass of ring is $\frac{2g}{3}$

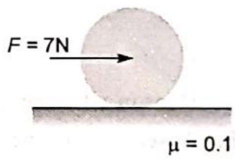
(ii) acceleration of hanging particle is $\frac{4g}{3}$

(iii) frictional force (on the ring) acts in forward direction

(iv) frictional force (on the ring) acts in backward direction

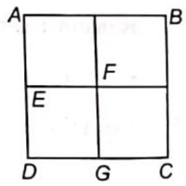
(a) statement (i) and (ii) only are correct
(b) statement (ii) and (iii) only are correct
(c) statement (iii) and (iv) only are correct
(d) None of these

**2.** A solid sphere of mass 10 kg is placed on rough surface having coefficient of friction $\mu = 0.1$. A constant force $F = 7$ N is applied along a line passing through the centre of the sphere as shown in the figure. The value of frictional force on the sphere is



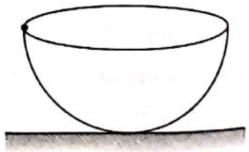

(a) 1 N
(b) 2 N
(c) 3 N
(d) 7 N

**3.** From a uniform square plate of side $a$ and mass $m$, a square portion $DEFG$ of side $\frac{a}{2}$ is removed. Then, the moment of inertia of remaining portion about the axis $AB$ is



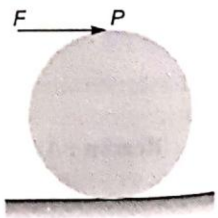

(a) $\frac{7ma^2}{16}$
(b) $\frac{3ma^2}{16}$
(c) $\frac{3ma^2}{4}$
(d) $\frac{9ma^2}{16}$

**4.** A small solid sphere of mass $m$ and radius $r$ starting from rest from the rim of a fixed hemispherical bowl of radius $R(>>r)$ rolls inside it without sliding. The normal reaction exerted by the sphere on the hemisphere when it reaches the bottom of hemisphere is



(a) $(3/7)\,mg$
(b) $(9/7)\,mg$
(c) $(13/7)\,mg$
(d) $(17/7)\,mg$

**5.** A uniform solid cylinder of mass $m$ and radius $R$ is placed on a rough horizontal surface. A horizontal constant force $F$ is applied at the top point $P$ of the cylinder so that it starts pure rolling. The acceleration of the cylinder is




(a) $F/3m$
(b) $2F/3m$
(c) $4F/3m$
(d) $5F/3m$

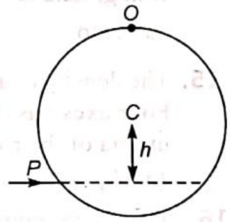
**6.** In the above question, the frictional force on the cylinder is

(a) $F/3$ towards right (b) $F/3$ towards left
(c) $2F/3$ towards right (d) $2F/3$ towards left

**7.** A small pulley of radius 20 cm and moment of inertia $0.32\ \text{kg-m}^2$ is used to hang a 2 kg mass with the help of massless string. If the block is released, for no slipping condition acceleration of the block will be

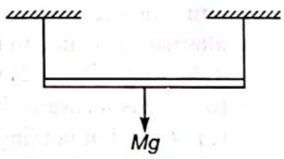

(a) $2\ \text{m/s}^2$
(b) $4\ \text{m/s}^2$
(c) $1\ \text{m/s}^2$
(d) $3\ \text{m/s}^2$

**8.** A uniform circular disc of radius $R$ is placed on a smooth horizontal surface with its plane horizontal and hinged at circumference through point $O$ as shown. An impulse $P$ is applied at a perpendicular distance $h$ from its centre $C$. The value of $h$ so that the impulse due to hinge is zero, is



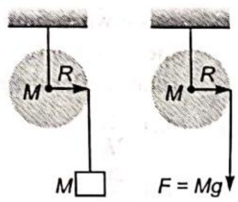

(a) $R$ (b) $R/2$
(c) $R/3$ (d) $R/4$

**9.** A rod is supported horizontally by means of two strings of equal length as shown in figure. If one of the string is cut. Then tension in other string at the same instant will



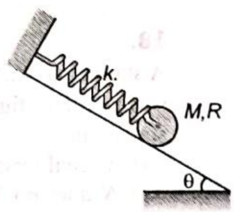

(a) remain unaffected
(b) increase
(c) decrease
(d) become equal to weight of the rod

**10.** The figure represents two cases. In first case a block of mass $M$ is attached to a string which is tightly wound on a disc of mass $M$ and radius $R$. In second case $F = Mg$. Initially, the disc is stationary in each case. If the same length of string is unwound from the disc, then




(a) same amount of work is done on both discs
(b) angular velocities of both the discs are equal
(c) both the discs have unequal angular accelerations
(d) All of the above

**11.** A uniform cylinder of mass $M$ and radius $R$ is released from rest on a rough inclined surface of inclination $\theta$ with the horizontal as shown in figure. As the cylinder rolls down the inclined surface, the maximum elongation in the spring of stiffness $k$ is

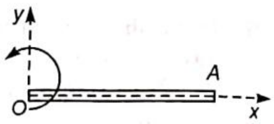



(a) $\dfrac{3}{4}\dfrac{Mg\sin\theta}{k}$ (b) $\dfrac{2Mg\sin\theta}{k}$
(c) $\dfrac{Mg\sin\theta}{k}$ (d) None of these

**12.** A uniform rod of mass $m$ and length $l$ rotates in a horizontal plane with an angular velocity $\omega$ about a vertical axis passing through one end. The tension in the rod at a distance $x$ from the axis is

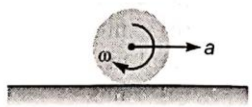
(a) $\frac{1}{2}m\omega^2 x$ (b) $\frac{1}{2}m\omega^2\left(1-\frac{x^2}{l}\right)$ (c) $\frac{1}{2}m\omega^2 l\left(1-\frac{x^2}{l^2}\right)$ (d) $\frac{1}{2}m\omega^2 l\left[1-\frac{x}{l}\right]$

**13.** A rod of length 1 m rotates in the $xy$ plane about the fixed point $O$ in the anticlockwise sense, as shown in figure with velocity $\omega = a + bt$ where $a = 10 \text{ rad s}^{-1}$ and $b = 5 \text{ rad s}^{-2}$. The velocity and acceleration of the point $A$ at $t = 0$ is




(a) $+10\hat{\mathbf{i}} \text{ ms}^{-1}$ and $+5\hat{\mathbf{i}} \text{ ms}^{-2}$
(b) $+10\hat{\mathbf{j}} \text{ ms}^{-1}$ and $(-100\hat{\mathbf{i}} + 5\hat{\mathbf{j}}) \text{ ms}^{-2}$
(c) $-10\hat{\mathbf{j}} \text{ ms}^{-1}$ and $(100\hat{\mathbf{i}} + 5\hat{\mathbf{j}}) \text{ ms}^{-2}$
(d) $-10\hat{\mathbf{j}} \text{ ms}^{-1}$ and $-5\hat{\mathbf{j}} \text{ ms}^{-2}$

**14.** A ring of radius $R$ rolls on a horizontal surface with constant acceleration $a$ of the centre of mass as shown in figure. If $\omega$ is the instantaneous angular velocity of the ring, then the net acceleration of the point of contact of the ring with ground is

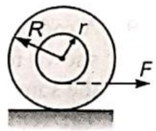



(a) zero
(b) $\omega^2 R$
(c) $a$
(d) $\sqrt{a^2 + (\omega^2 R)^2}$

**15.** The density of a rod $AB$ increases linearly from $A$ to $B$. Its midpoint is $O$ and its centre of mass is at $C$. Four axes pass through $A, B, O$ and $C$, all perpendicular to the length of the rod. The moments of inertia of the rod about these axes are $I_A, I_B, I_O$ and $I_C$ respectively. Then

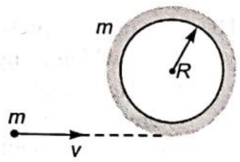
(a) $I_A > I_B$
(b) $I_C < I_B$
(c) $I_O > I_C$
(d) All of these

**16.** The figure shows a spool placed at rest on a horizontal rough surface. A tightly would string on the inner cylinder is pulled horizontally with a force $F$. Identity the correct alternative related to the friction force $f$ acting on the spool



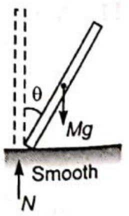

(a) $f$ acts leftwards with $f < F$
(b) $f$ acts leftwards but nothing can be said about its magnitude
(c) $f < F$ but nothing can be said about its magnitude
(d) None of the above

**17.** A circular ring of mass $m$ and radius $R$ rests flat on a horizontal smooth surface as shown in figure. A particle of mass $m$, and moving with a velocity $v$, collides inelastically $(e = 0)$ with the ring. The angular velocity with which the system rotates after the particle strikes the ring is




(a) $\dfrac{v}{2R}$
(b) $\dfrac{v}{3R}$
(c) $\dfrac{2v}{3R}$
(d) $\dfrac{3v}{4R}$

**18.**
A stationary uniform rod in the upright position is allowed to fall on a smooth horizontal surface. The figure shows the instantaneous position of the rod. Identity the correct statement.

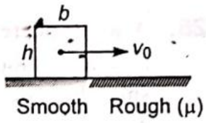



(a) normal reaction $N$ is equal to $Mg$
(b) $N$ does positive rotational work about the centre of mass
(c) a couple of equal and opposite forces acts on the rod
(d) All of the above

**19.** A thin uniform rod of mass $m$ and length $l$ is free to rotate about its upper end. When it is at rest. It receives an impulse $J$ at its lowest point, normal to its length. Immediately after impact

(a) the angular momentum of the rod is $Jl$
(b) the angular velocity of the rod is $3J/ml$
(c) the kinetic energy of the rod is $3J^2/2m$
(d) All of the above

**20.** A rectangular block of size $(b \times h)$ moving with velocity $v_o$ enters on a rough surface where the coefficient of friction is $\mu$ as shown in figure. Identify the correct statement.



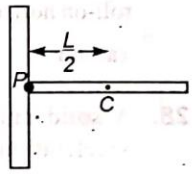

(a) The net torque acting on the block about its COM is $\mu\, mg \dfrac{h}{2}$ (clockwise)

(b) The net torque acting on the block about its COM is zero

(c) The net torque acting on the block about its COM is in the anticlockwise sense

(d) None of the above

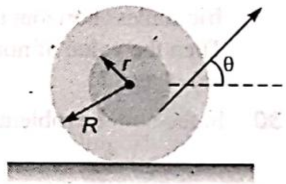
**21.** A uniform rod of length $L$ and mass $m$ is free to rotate about a frictionless pivot at one end as shown in figure. The rod is held at rest in the horizontal position and a coin of mass $m$ is placed at the free end. Now the rod is released. The reaction on the coin immediately after the rod starts falling is




(a) $\dfrac{3\,mg}{2}$ (b) $2\,mg$

(c) zero (d) $\dfrac{mg}{2}$

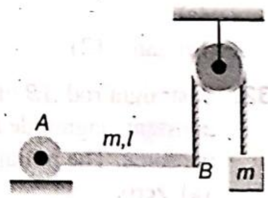
**22.** A spool is pulled at an angle $\theta$ with the horizontal on a rough horizontal surface as shown in the figure. If the spool remains at rest, the angle $\theta$ is equal to




(a) $\cos^{-1}\left(\dfrac{R}{r}\right)$ (b) $\sin^{-1}\left(\sqrt{1-\dfrac{r^2}{R^2}}\right)$

(c) $\pi-\cos^{-1}\left(\dfrac{r}{R}\right)$ (d) $\sin^{-1}\left(\dfrac{r}{R}\right)$

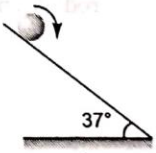
**23.** Uniform rod $AB$ is hinged at end $A$ in horizontal position as shown in the figure. The other end is connected to a block through a massless string as shown. The pulley is smooth and massless. Mass of block and rod is same and is equal to $m$. Then acceleration of block just after release from this position is




(a) $6g/13$ (b) $g/4$

(c) $3g/8$ (d) None of these

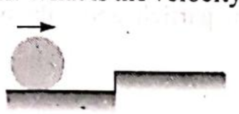
**24.** A cylinder having radius 0.4 m, initially rotating (at $t = 0$) with $\omega_0 = 54$ rad/sec is placed on a rough inclined plane with $\theta = 37°$ having friction coefficient $\mu = 0.5$. The time taken by the cylinder to start pure rolling is $(g = 10\ \text{m/s}^2)$




(a) 5.4 s (b) 1.2 s

(c) 1.4 s (d) 1.8 s

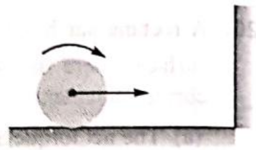
**25.** A disc of mass $M$ and radius $R$ is rolling purely with center's velocity $v_0$ on a flat horizontal floor when it hits a step in the floor of height $R/4$. The corner of the step is sufficiently rough to prevent any slipping of the disc against itself. What is the velocity of the centre of the disc just after impact?



(a) $4v_0/5$ (b) $4v_0/7$

(c) $5v_0/6$ (d) None of these

**26.** A solid sphere is rolling purely on a rough horizontal surface (coefficient of kinetic friction $=\mu$) with speed of centre $= u$. It collides inelastically with a smooth vertical wall at a certain moment, the coefficient of restitution being $\frac{1}{2}$. The sphere will begin pure rolling after a time



(a) $\frac{3u}{7\mu g}$ (b) $\frac{2u}{7\mu g}$ (c) $\frac{3u}{5\mu g}$ (d) $\frac{2u}{5\mu g}$

**27.** A thin hollow sphere of mass $m$ is completely filled with non viscous liquid of mass $m$. When the sphere roll-on horizontal ground such that centre moves with velocity $v$, kinetic energy of the system is equal to

(a) $mv^2$ (b) $\frac{4}{3}mv^2$ (c) $\frac{4}{5}mv^2$ (d) None of these

**28.** A solid uniform disc of mass $m$ rolls without slipping down a fixed inclined plank with an acceleration $a$. The frictional force on the disc due to surface of the plane is

(a) $\frac{1}{4}ma$ (b) $\frac{3}{2}ma$ (c) $ma$ (d) $\frac{1}{2}ma$

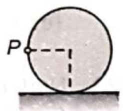
**29.** A uniform slender rod of mass $m$ and length $L$ is released from rest, with its lower end touching a frictionless horizontal floor. At the initial moment, the rod is inclined at an angle $\theta = 30°$ with the vertical. Then the value of normal reaction from the floor just after release will be

(a) $4mg/7$ (b) $5mg/9$ (c) $2mg/5$ (d) None of these

**30.** In the above problem, the initial acceleration of the lower end of the rod will be

(a) $g\sqrt{3}/4$ (b) $g\sqrt{3}/5$ (c) $3g\sqrt{3}/7$ (d) None of these

**31.** A disc of radius $R$ is rolling purely on a flat horizontal surface, with a constant angular velocity. The angle between the velocity and acceleration vectors of point $P$ is

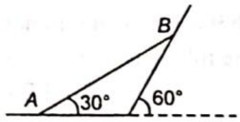



(a) zero (b) 45°
(c) $\tan^{-1}(2)$ (d) $\tan^{-1}(1/2)$

**32.** A straight rod $AB$ of mass $M$ and length $L$ is placed on a frictionless horizontal surface. A force having constant magnitude $F$ and a fixed direction starts acting at the end $A$. The rod is initially perpendicular to the force. The initial acceleration of end $B$ is

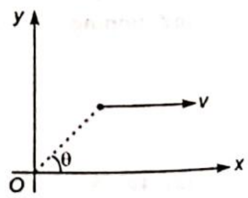
(a) zero (b) $2F/M$ (c) $4F/M$ (d) None of these

**33.** In the figure shown, the instantaneous speed of end $A$ of the rod is $v$ to the left. The angular velocity of the rod of length $L$, must be




(a) $v/2L$ (b) $v/L$ (c) $\frac{\sqrt{3}v}{2L}$ (d) $\frac{2v}{L}$

**34.** A particle moves parallel to $x$-axis with constant velocity $v$ as shown in the figure. The angular velocity of the particle about the origin $O$

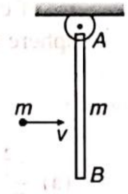



(a) remains constant
(b) continuously increases
(c) continuously decreases
(d) oscillates

**35.** A thin uniform rod of mass $M$ and length $L$ is hinged at its upper end, and released from rest from a horizontal position. The tension at a point located at a distance $L/3$ from the hinge point, when the rod becomes vertical, will be

(a) $22\,Mg/27$ (b) $11\,Mg/13$ (c) $6\,Mg/11$ (d) $2\,Mg$

**36.** A uniform rod $AB$ of length $L$ and mass $m$ is suspended freely at $A$ and hangs vertically at rest when a particle of same mass $m$ is fired horizontally with speed $v$ to strike the rod at its mid point. If the particle is brought to rest after the impact. Then the impulsive reaction at $A$ in horizontal direction is



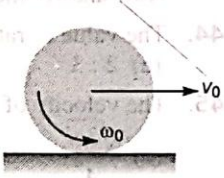

(a) $mv/4$ (b) $mv/2$
(c) $mv$ (d) $2\,mv$

**37.** A child with mass $m$ is standing at the edge of a merry go round having moment of inertia $I$, radius $R$ and initial angular velocity $\omega$ as shown in the figure. The child jumps off the edge of the merry go round with tangential velocity $v$ with respect to the ground. The new angular velocity of the merry go round is


(a) $\sqrt{\dfrac{I\omega^2 - mv^2}{I}}$ (b) $\sqrt{\dfrac{(I + mR^2)\omega^2 - mv^2}{I}}$

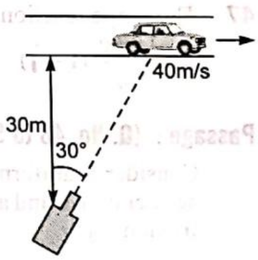
(c) $\dfrac{I\omega - mvR}{I}$ (d) $\dfrac{(I + mR^2)\,\omega - mvR}{I}$

**38.** A disc of radius $R$ is spun to an angular speed $\omega_0$ about its axis and then imparted a horizontal velocity of magnitude $\dfrac{\omega_0 R}{4}$. The coefficient of friction is $\mu$. The sense of rotation and direction of linear velocity are shown in the figure. The disc will return to its initial position



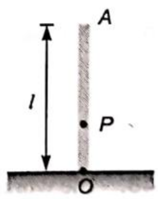

(a) if the value of $\mu < 0.5$ (b) irrespective of the value of $\mu$
(c) if the value of $0.5 < \mu < 1$ (d) if $\mu > 1$

**39.** A racing car is travelling along a straight track at a constant velocity of 40 m/s. A fixed TV camera is recording the event as shown in figure. In order to keep the car in view, in the position shown, the angular velocity of camera should be




(a) 3 rad/s
(b) 2 rad/s
(c) 4 rad/s
(d) 1 rad/s

**40.** A uniform rod $OA$ of length $l$, resting on smooth surface is slightly distributed from its vertical position. $P$ is a point on the rod whose locus is a circle during the subsequent motion of the rod. Then the distance $OP$ is equal to



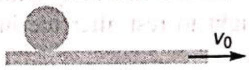

(a) $l/2$
(b) $l/3$
(c) $l/4$
(d) there is no such point

**41.** In the above question, the velocity of end $O$ when end $A$ hits the ground is
(a) zero
(b) along the horizontal
(c) along the vertical
(d) at some inclination to the ground ($\neq 90°$)

**42.** In the above question, the velocity of end $A$ at the instant it hits the ground is

(a) $\sqrt{3gl}$ (b) $\sqrt{12gl}$ (c) $\sqrt{6gl}$ (d) None of these

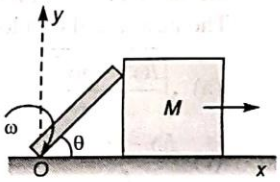
**43.** A solid sphere of mass $m$ and radius $R$ is gently placed on a conveyer belt moving with constant velocity $v_0$. If coefficient of friction between belt and sphere is 2/7, the distance traveled by the centre of the sphere before it starts pure rolling is




(a) $\frac{v_0^2}{7g}$ (b) $\frac{2v_0^2}{49g}$ (c) $\frac{2v_0^2}{5g}$ (d) $\frac{2v_0^2}{7g}$

**Passage : (Q. No. 44 to 47)**

A uniform rod of mass $m$ and length $l$ is pivoted at point $O$. The rod is initially in vertical position and touching a block of mass $M$ which is at rest on a horizontal surface. The rod is given a slight jerk and it starts rotating about point $O$. This causes the block to move forward as shown. The rod loses contact with the block at $\theta = 30°$. All surfaces are smooth. Now answer the following questions.




**44.** The value of ratio $M/m$ is

(a) 2 : 3 (b) 3 : 2 (c) 4 : 3 (d) 3 : 4

**45.** The velocity of block when the rod loses contact with the block is

(a) $\frac{\sqrt{3gl}}{4}$ (b) $\frac{\sqrt{5gl}}{4}$ (c) $\frac{\sqrt{6gl}}{4}$ (d) $\frac{\sqrt{7gl}}{4}$

**46.** The acceleration of centre of mass of rod, when it loses contact with the block is

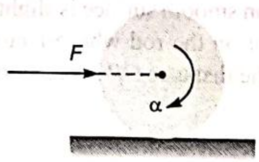
(a) $5g/4$ (b) $5g/2$ (c) $3g/2$ (d) $3g/4$

**47.** The hinge reaction at $O$ on the rod when it loses contact with the block is

(a) $\frac{3mg}{4}(\hat{i}+\hat{j})$ (b) $\left(\frac{mg}{4}\right)\hat{j}$ (c) $\left(\frac{mg}{4}\right)\hat{i}$ (d) $\frac{mg}{4}(\hat{i}+\hat{j})$

**Passage : (Q. No. 48 to 50)**

Consider a uniform disc of mass $m$, radius $r$, rolling without slipping on a rough surface with linear acceleration a and angular acceleration $\alpha$ due to an external force $F$ as shown in the figure. Coefficient of friction is $\mu$




**48.** The work done by the frictional force at the instant of pure rolling is

(a) $\frac{\mu mgat^2}{2}$ (b) $\mu mgat^2$ (c) $\mu mg\frac{at^2}{\alpha}$ (d) zero

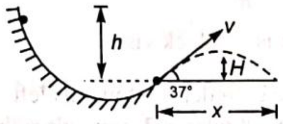
**49.** The magnitude of frictional force acting on the disc is

(a) $ma$ (b) $\mu mg$ (c) $\frac{ma}{2}$ (d) zero

**50.** Angular momentum of the disc will be conserved about
(a) centre of mass
(b) point of contact
(c) a point at a distance $3R/2$ vertically above the point of contact
(d) a point at a distance $4R/3$ vertically above the point of contact

**Passage : (Q. No. 51 to 53)**

A tennis ball, starting from rest, rolls down the hill in the drawing. At the end of the hill the ball becomes airborne, leaving at an angle of 37° with respect to the ground. Treat the ball as a thin-walled spherical shell.




**51.** The velocity of projection $v$ is
(a) $\sqrt{2gh}$ (b) $\sqrt{\frac{10}{7}gh}$ (c) $\sqrt{\frac{5}{7}gh}$ (d) $\sqrt{\frac{6}{5}gh}$

**52.** Maximum height reached by ball $H$ above ground is
(a) $\frac{9h}{35}$ (b) $\frac{18h}{35}$ (c) $\frac{18h}{25}$ (d) $\frac{27h}{125}$

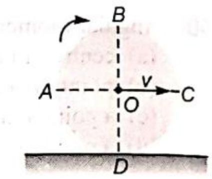
**53.** Range $x$ of the ball is
(a) $\frac{144}{125}h$ (b) $\frac{48}{25}h$ (c) $\frac{48}{35}h$ (d) $\frac{24}{7}h$

## More than One Correct Options

**1.** A mass $m$ of radius $r$ is rolling horizontally without any slip with a linear speed $v$. It then rolls up to a height given by $\frac{3}{4}\frac{v^2}{g}$
(a) the body is identified to be a disc or a solid cylinder
(b) the body is a solid sphere
(c) moment of inertia of the body about instantaneous axis of rotation is $\frac{3}{2}mr^2$
(d) moment of inertia of the body about instantaneous axis of rotation is $\frac{7}{5}mr^2$

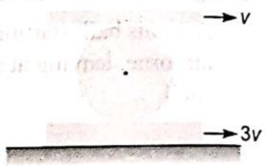
**2.** Four identical rods each of mass $m$ and length $l$ are joined to form a rigid square frame. The frame lies in the $xy$ plane, with its centre at the origin and the sides parallel to the $x$ and $y$ axes. Its moment of inertia about
(a) the $x$-axis is $\frac{2}{3}ml^2$
(b) the $z$-axis is $\frac{4}{3}ml^2$
(c) an axis parallel to the $z$-axis and passing through a corner is $\frac{10}{3}ml^2$
(d) one side is $\frac{5}{3}ml^2$

**3.** A uniform circular ring rolls without slipping on a horizontal surface. At any instant, its position is as shown in the figure. Then



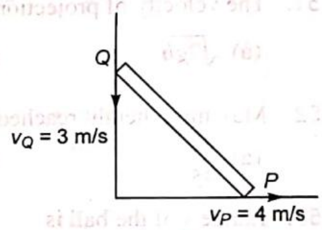

(a) section $ABC$ has greater kinetic energy than section $ADC$
(b) section $BC$ has greater kinetic energy than section $CD$
(c) section $BC$ has the same kinetic energy as section $DA$
(d) the sections $CD$ and $DA$ have the same kinetic energy

**4.** A cylinder of radius $R$ is to roll without slipping between two planks as shown in the figure. Then



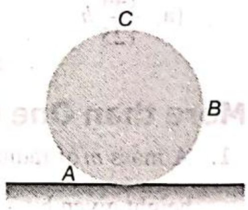

(a) angular velocity of the cylinder is $\frac{v}{R}$ counter clockwise
(b) angular velocity of the cylinder is $\frac{2v}{R}$ clockwise
(c) velocity of centre of mass of the cylinder is $v$ towards left
(d) velocity of centre of mass of the cylinder is $2v$ towards right

**5.** A uniform rod of mass $m = 2$ kg and length $l = 0.5$ m is sliding along two mutually perpendicular smooth walls with the two ends $P$ and $Q$ having velocities $v_P = 4$ m/s and $v_Q = 3$ m/s as shown. Then




(a) The angular velocity of rod, $\omega = 10$ rad/s, counter clockwise
(b) The angular velocity of rod, $\omega = 5.0$ rad/s, counter clockwise
(c) The velocity of centre of mass of rod, $v_{cm} = 2.5$ m/s
(d) The total kinetic energy of rod, $K = \frac{25}{3}$ joule

**6.** A wheel is rolling without slipping on a horizontal plane with velocity $v$ and acceleration $a$ of centre of mass as shown in figure. Acceleration at




(a) $A$ is vertically upwards
(b) $B$ may be vertically downwards
(c) $C$ cannot be horizontal
(d) a point on the rim may be horizontal leftwards

**7.** A uniform rod of length $l$ and mass $2m$ rests on a smooth horizontal table. A point mass $m$ moving horizontally at right angles to the rod with velocity $v$ collides with one end of the rod and sticks it. Then

(a) angular velocity of the system after collision is $\frac{v}{l}$
(b) angular velocity of the system after collision is $\frac{v}{2l}$
(c) the loss in kinetic energy of the system as a whole as a result of the collision is $\frac{mv^2}{6}$
(d) the loss in kinetic energy of the system as a whole as a result of the collision is $\frac{7mv^2}{24}$

**8.** A nonuniform ball of radius $R$ and radius of gyration about geometric centre $= R/2$, is kept on a frictionless surface. The geometric centre coincides with the centre of mass. The ball is struck horizontally with a sharp impulse $= J$. The point of application of the impulse is at a height $h$ above the surface. Then

(a) the ball with slip on surface for all cases
(b) the ball will roll purely if $h = 5R/4$
(c) the ball will roll purely if $h = 3R/2$
(d) there will be no rotation if $h = R$

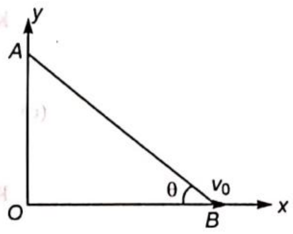
**9.** A hollow spherical ball is given an initial push up an incline of inclination angle $\alpha$. The ball rolls purely. Coefficient of static friction between ball and incline $= \mu$. During its upwards journey

(a) friction acts up along the incline
(b) $\mu \geq 2 \tan \alpha/5$
(c) friction acts down along the incline
(d) $\mu \geq 2 \tan \alpha/7$

**10.** A uniform disc of mass $m$ and radius $R$ rotates about a fixed vertical axis passing through its centre with angular velocity $\omega$. A particle of same mass m and having velocity of $2\omega R$ towards centre of the disc collides with the disc moving horizontally and sticks to its rim.

(a) The angular velocity of the disc will become $\omega/3$
(b) The angular velocity of the disc will become $5\omega/3$
(c) The impulse on the particle due to disc is $\frac{\sqrt{37}}{3} m\omega R$
(d) The impulse on the particle due to disc is $2m\omega R$

**11.** The end $B$ of the rod $AB$ which makes angle $\theta$ with the floor is being pulled with a constant velocity $v_0$ as shown. The length of the rod is $l$.



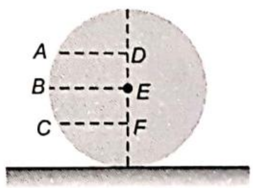

(a) At $\theta = 37°$ velocity of end $A$ is $\frac{4}{3} v_0$ downwards
(b) At $\theta = 37°$ angular velocity of rod is $\frac{5v_0}{3l}$
(c) Angular velocity of rod is constant
(d) Velocity of end $A$ is constant

## Match the Columns

**1.** A solid sphere, a hollow sphere and a disc of same mass and same radius are released from a rough inclined plane. All of them rolls down without slipping. On reaching the bottom of the plane, match the two columns.

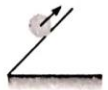
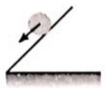
| Column I | Column II |
|---|---|
| (a) time taken to reach the bottom | (p) maximum for solid sphere |
| (b) total kinetic energy | (q) maximum for hollow sphere |
| (c) rotational kinetic energy | (r) maximum for disc |
| (d) translational kinetic energy | (s) same for all |

**2.** A solid sphere is placed on a rough ground as shown. $E$ is the centre of sphere and $DE > EF$. We have to apply a linear impulse either at point $A$, $B$ or $C$. Match the following two columns.

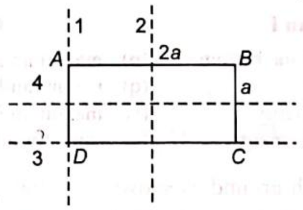



| Column I | Column II |
|---|---|
| (a) Sphere will acquire maximum angular speed it impulse is applied at | (p) A |
| (b) Sphere will acquire maximum linear speed it impulse is applied at | (q) B |
| (c) Sphere can roll without slipping if impulse is applied at | (r) C |
| (d) Sphere can roll with forward slipping if impulse is applied at | (s) at any point A, B or C |

**3.** The inclined surfaces shown in column I are sufficiently rough. In column II direction and magnitudes of frictional forces are mentioned. Match the two columns.

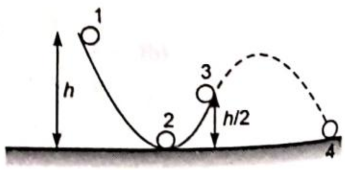
| Column I | Column II |
|---|---|
| (a)  Rolling upwards | (p) upwards |
| (b)  Kept in rotating position | (q) downwards |
| (c)  Kept in translational position | (r) maximum friction will act |
| (d)  Kept in translational position | (s) required value of friction will act |

**4.** A rectangular slab *ABCD* have dimensions $a \times 2a$ as shown in figure. Match the following two columns.



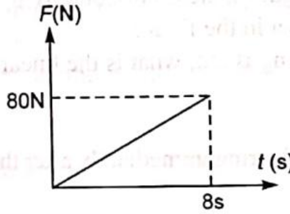

| Column I | Column II |
|---|---|
| (a) Radius of gyration about axis-1 | (p) $\frac{a}{\sqrt{12}}$ |
| (b) Radius of gyration about axis-2 | (q) $\frac{2a}{\sqrt{3}}$ |
| (c) Radius of gyration about axis-3 | (r) $\frac{a}{\sqrt{3}}$ |
| (d) Radius of gyration about axis-4 | (s) None |

**5.** A small solid ball rolls down along sufficiently rough surface from 1 to 3 as shown in figure. From point-3 onwards it moves under gravity. Match the following two columns.



| Column I | Column II |
|---|---|
| (a) Rotational kinetic energy of ball at point-2 | (p) $\frac{1}{7}\,mgh$ |
| (b) Translational kinetic energy of ball at point-3 | (q) $\frac{2}{7}\,mgh$ |
| (c) Rotational kinetic energy of ball at point-4 | (r) $\frac{5}{7}\,mgh$ |
| (d) Translational kinetic energy of ball at point-4 | (s) None |

**6.** A uniform disc of mass 10 kg, radius 1 m is placed on a rough horizontal surface. The co-efficient of friction between the disc and the surface is 0.2. A horizontal time varying force is applied on the centre of the disc whose variation with time is shown in graph.



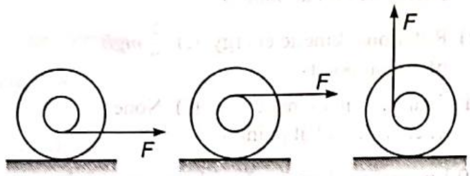

| Column I | Column II |
|---|---|
| (a) Disc rolls without slipping | (p) at $t = 7$ s |
| (b) Disc rolls with slipping | (q) at $t = 3$ s |
| (c) Disc starts slipping at | (r) at $t = 4$ s |
| (d) Friction force is 10 N at | (s) None |

**7.** Match the columns.

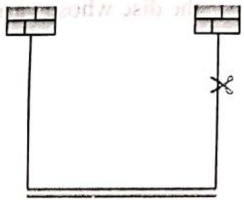
| Column I | Column II |
|---|---|
| (a) Moment of inertia of a circular disc of mass $M$ and radius $R$ about a tangent parallel to plane of disc | (p) $\frac{MR^2}{2}$ |
| (b) Moment of inertia of a solid sphere of mass $M$ and radius $R$ about a tangent | (q) $\frac{7}{5}MR^2$ |
| (c) Moment of inertia of a circular disc of mass $M$ and radius $R$ about a tangent perpendicular to plane of disc | (r) $\frac{5}{4}MR^2$ |
| (d) Moment of inertia of a cylinder of mass $M$ and radius $R$ about its axis | (s) $\frac{3}{2}MR^2$ |

## Subjective Questions (Level 2)

1. Figure shows three identical yo-yos initially at rest on a horizontal surface. For each yo-yo the string is pulled in the direction shown. In each case there is sufficient friction for the yo-yo to roll without slipping. Draw the free-body diagram for each yo-yo. In what direction will each yo-yo rotate ?



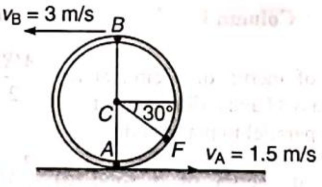

2. A uniform rod of mass $m$ and length $l$ is held horizontally by two vertical strings of negligible mass, as shown in the figure.
   (a) Immediately after the right string is cut, what is the linear acceleration of the free end of the rod ?
   (b) Of the middle of the rod ?
   (c) Determine the tension in the left string immediately after the right string is cut.



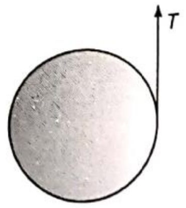
3. A solid disk is rolling without slipping on a level surface at a constant speed of 2.00 m/s. How far can it roll up a 30° ramp before it stops? (Take $g = 9.8\,\text{m/s}^2$)

4. A lawn roller in the form of a thin-walled hollow cylinder of mass $M$ is pulled horizontally with a constant horizontal force $F$ applied by a handle attached to the axle. If it rolls without slipping, find the acceleration and the friction force.

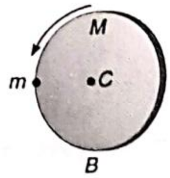
5. Due to slipping, points $A$ and $B$ on the rim of the disk have the velocities shown. Determine the velocities of the centre point $C$ and point $F$ at this instant.




6. A uniform cylinder of mass $M$ and radius $R$ has a string wrapped around it. The string is held fixed and the cylinder falls vertically, as in figure.
   (a) Show that the acceleration of the cylinder is downward with magnitude $a = \frac{2g}{3}$.
   (b) Find the tension in the string.

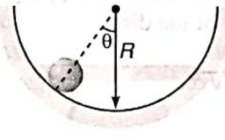



7. A uniform disc of mass $M$ and radius $R$ is pivoted about the horizontal axis through its centre $C$. A point mass $m$ is glued to the disc at its rim, as shown in figure. If the system is released from rest, find the angular velocity of the disc when $m$ reaches the bottom point $B$.

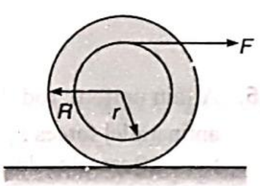


**8.** A disc of radius $R$ and mass $m$ is projected on to a horizontal floor with a backward spin such that its centre of mass speed is $v_0$ and angular velocity is $\omega_0$. What must be the minimum value of $\omega_0$ so that the disc eventually returns back?

**9.** A ball of mass $m$ and radius $r$ rolls along a circular path of radius $R$. Its speed at the bottom ($\theta = 0°$) of the path is $v_0$. Find the force of the path on the ball as a function of $\theta$.

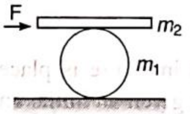



**10.** A heavy homogeneous cylinder has mass $m$ and radius $R$. It is accelerated by a force $F$, which is applied through a rope wound around a light drum of radius $r$ attached to the cylinder (figure). The coefficient of static friction is sufficient for the cylinder to roll without slipping.



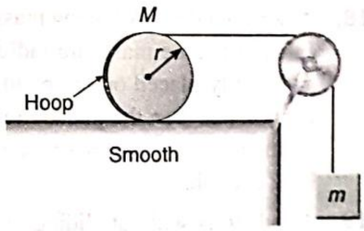

(a) Find the friction force.
(b) Find the acceleration $a$ of the centre of the cylinder.
(c) Is it possible to choose $r$, so that $a$ is greater than $\frac{F}{m}$? How ?
(d) What is the direction of the friction force in the circumstances of part (c) ?

**11.** A man pushes a cylinder of mass $m_1$ with the help of a plank of mass $m_2$ as shown. There is no slipping at any contact. The horizontal component of the force applied by the man is $F$. Find :



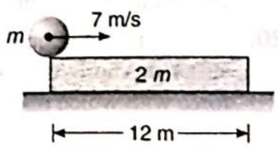

(a) the acceleration of the plank and the centre of mass of the cylinder and
(b) the magnitudes and directions of frictional forces at contact points.

**12.** For the system shown in figure, $M = 1\,\text{kg}$, $m = 0.2\,\text{kg}$, $r = 0.2\,\text{m}$.
Calculate : ($g = 10\,\text{m/s}^2$)
(a) the linear acceleration of hoop,
(b) the angular acceleration of the hoop of mass $M$ and
(c) the tension in the rope.

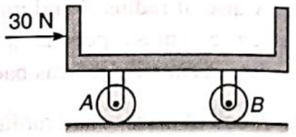



**Note** *Treat hoop as the ring. Assume no slipping between string and hoop.*

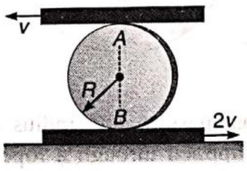
**13.** A cylinder of mass $m$ is kept on the edge of a plank of mass $2m$ and length 12 m, which in turn is kept on smooth ground. Coefficient of friction between the plank and the cylinder is 0.1. The cylinder is given an impulse, which imparts it a velocity 7 m/s but no angular velocity. Find the time after which the cylinder falls off the plank. ($g = 10\,\text{m/s}^2$)



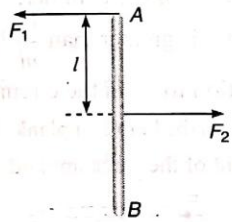
**14.** The 9 kg cradle is supported as shown by two uniform disks that roll without sliding at all surfaces of contact. The mass of each disk is $m = 6\,\text{kg}$ and the radius of each disk is $r = 80\,\text{mm}$. Knowing that the system is initially at rest, determine the velocity of the cradle after it has moved 250 mm.



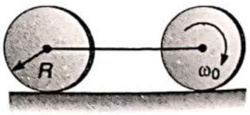

**15.** The disc of the radius $R$ is confined to roll without slipping at $A$ and $B$. If the plates have the velocities shown, determine the angular velocity of the disc.



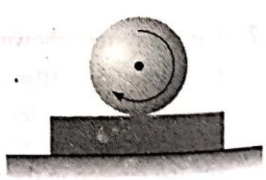

**16.** A thin uniform rod $AB$ of mass $m = 1\,\text{kg}$ moves translationally with acceleration $a = 2\,\text{m/s}^2$ due to two antiparallel forces $F_1$ and $F_2$. The distance between the points at which these forces are applied is equal to $l = 20\,\text{cm}$. Besides, it is known that $F_2 = 5\,\text{N}$. Find the length of the rod.




**17.** The assembly of two discs as shown in figure is placed on a rough horizontal surface and the front disc is given an initial angular velocity $\omega_0$. Determine the final linear and angular velocity when both the discs start rolling. It is given that friction is sufficient to sustain rolling in the rear wheel from the starting of motion.




**18.** A horizontal plank having mass $m$ lies on a smooth horizontal surface. A sphere of same mass and radius $r$ is spined to an angular frequency $\omega_0$ and gently placed on the plank as shown in the figure. If coefficient of friction between the plank and the sphere is $\mu$. Find the distance moved by the plank till the sphere starts pure rolling on the plank. The plank is long enough.



**19.** A ball rolls without sliding over a rough horizontal floor with velocity $v_0 = 7$ m/s towards a smooth vertical wall. If coefficient of restitution between the wall and the ball is $e = 07$. Calculate velocity $v$ of the ball long after the collision.

**20.** A uniform rod of mass $m$ and length $l$ rests on a smooth horizontal surface. One of the ends of the rod is struck in a horizontal direction at right angles to the rod. As a result the rod obtains velocity $v_0$. Find the force with which one-half of the rod will act on the other in the process of motion.

**21.** A sphere, a disk and a hoop made of homogeneous materials have the same radius (10 cm) and mass (3 kg). They are released from rest at the top of a 30° incline and roll down without slipping through a vertical distance of 2 m. ($g = 9.8\,\text{m/s}^2$)

(a) What are their speeds at the bottom ?
(b) Find the frictional force $f$ in each case
(c) If they start together at $t = 0$, at what time does each reach the bottom ?

**22.** $ABC$ is a triangular framework of three uniform rods each of mass $m$ and length $2l$. It is free to rotate in its own plane about a smooth horizontal axis through $A$ which is perpendicular to $ABC$. If it is released from rest when $AB$ is horizontal and $C$ is above $AB$. Find the maximum velocity of $C$ in the subsequent motion.

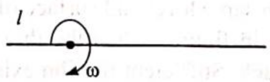
**23.** A uniform stick of length $L$ and mass $M$ hinged at one end is released from rest at an angle $\theta_0$ with the vertical. Show that when the angle with the vertical is $\theta$, the hinge exerts a force $F_r$ along the stick and $F_t$ perpendicular to the stick given by

$$F_r = \frac{1}{2} Mg\,(5\cos\theta - 3\cos\theta_0) \quad \text{and} \quad F_t = \frac{1}{4} Mg \sin\theta$$

**24.** A uniform rod $AB$ of mass $3m$ and length $4l$, which is free to turn in a vertical plane about a smooth horizontal axis through $A$, is released from rest when horizontal. When the rod first becomes vertical, a point $C$ of the rod, where $AC = 3l$, strikes a fixed peg. Find the linear impulse exerted by the peg on the rod if :

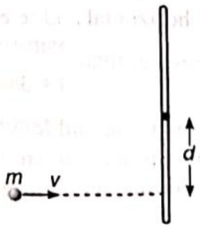
(a) the rod is brought to rest by the peg,
(b) the rod rebounds and next comes to instantaneous rest inclined to the downward vertical at an angle $\frac{\pi}{3}$ radian.

**25.** A uniform rod of length $4l$ and mass $m$ is free to rotate about a horizontal axis passing through a point distant $l$ from its one end. When the rod is horizontal, its angular velocity is $\omega$ as shown in figure. Calculate :

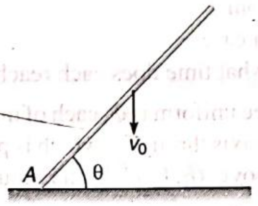



(a) reaction of axis at this instant,
(b) acceleration of centre of mass of the rod at this instant,
(c) reaction of axis and acceleration of centre mass of the rod when rod becomes vertical for the first time,
(d) minimum value of $\omega$, so that centre of rod can complete circular motion.

**26.** A stick of length $l$ lies on horizontal table. It has a mass $M$ and is free to move in any way on the table. A ball of mass $m$, moving perpendicularly to the stick at a distance $d$ from its centre with speed $v$ collides elastically with it as shown in figure. What quantities are conserved in the collision ? What must be the mass of the ball, so that it remains at rest immediately after collision?



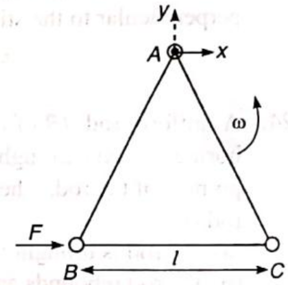
**27.** A rod of length $l$ forming an angle $\theta$ with the horizontal strikes a frictionless floor at $A$ with its centre of mass velocity $v_0$ and no angular velocity. Assuming that the impact at $A$ is perfectly elastic. Find the angular velocity of the rod immediately after the impact.




**28.** Three particles $A$, $B$ and $C$, each of mass $m$, are connected to each other by three massless rigid rods to form a rigid, equilateral triangular body of side $l$. This body is placed on a horizontal frictionless table ($x$-$y$ plane) and is hinged to it at the point $A$, so that it can move without friction about the vertical axis through $A$ (see figure). The body is set into rotational motion on the table about $A$ with a constant angular velocity $\omega$.

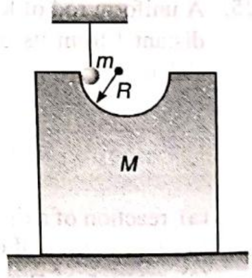
(a) Find the magnitude of the horizontal force exerted by the hinge on the body.

(b) At time $T$, when the side $BC$ is parallel to the $x$-axis, a force $F$ is applied on $B$ along $BC$ (as shown). Obtain the $x$-component and the $y$-component of the force exerted by the hinge on the body, immediately after time $T$.




**29.** A semicircular track of radius $R = 62.5\,\text{cm}$ is cut in a block. Mass of block, having track, is $M = 1$ kg and rests over a smooth horizontal floor. A cylinder of radius $r = 10\,\text{cm}$ and mass $m = 0.5\,\text{kg}$ is hanging by thread such that axes of cylinder and track are in same level and surface of cylinder is in contact with the track as shown in figure. When the thread is burnt, cylinder starts to move down the track. Sufficient friction exists between surface of cylinder and track, so that cylinder does not slip.

Calculate velocity of axis of cylinder and velocity of the block when it reaches bottom of the track. Also find force applied by block on the floor at that moment. ($g = 10\,\text{m/s}^2$)

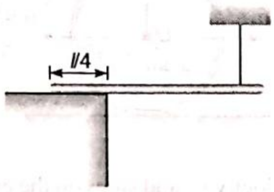



**30.** A uniform circular cylinder of mass $m$ and radius $r$ is given an initial angular velocity $\omega_0$ and no initial translational velocity. It is placed in contact with a plane inclined at an angle $\alpha$ to the horizontal. If there is a coefficient of friction $\mu$ for sliding between the cylinder and plane. Find the distance the cylinder moves up before sliding stops. Also, calculate the maximum distance it travels up the plane. Assume $\mu > \tan\alpha$.

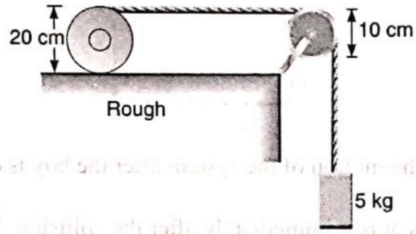
**31.** Show that if a rod held at angle $\theta$ to the horizontal and released, its lower end will not slip if the friction coefficient between rod and ground is greater than $\dfrac{3\sin\theta\cos\theta}{1+3\sin^2\theta}$.

**32.** One-fourth length of a uniform rod of mass $m$ and length $l$ is placed on a rough horizontal surface and it is held stationary in horizontal position by means of a light thread as shown in the figure. The thread is then burnt and the rod starts rotating about the edge. Find the angle between the rod and the

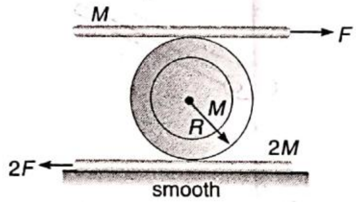
horizontal when it is about to slide on the edge. The coefficient of friction between the rod and the surface is $\mu$.



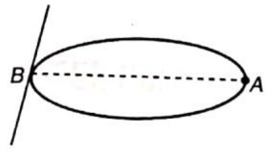

**33.** In figure the cylinder of mass 10 kg and radius 10 cm has a tape wrapped round it. The pulley weighs 100 N and has a radius 5 cm. When the system is released, the 5 kg mass comes down and the cylinder rolls without slipping. Calculate the acceleration and velocity of the mass as a function of time.



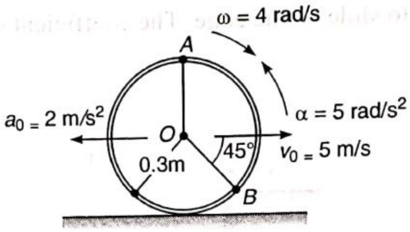

**34.** A cylinder is sandwiched between two planks. Two constant horizontal forces $F$ and $2F$ are applied on the planks as shown. Determine the acceleration of the centre of mass of cylinder and the top plank, if there is no slipping at the top and bottom of cylinder.



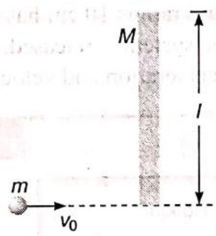

**35.** A ring of mass $m$ and radius $r$ has a particle of mass $m$ attached to it at a point $A$. The ring can rotate about a smooth horizontal axis which is tangential to the ring at a point $B$ diametrically opposite to $A$. The ring is released from rest when $AB$ is horizontal. Find the angular velocity and the angular acceleration of the body when $AB$ has turned through an angle $\frac{\pi}{3}$.




**36.** A hoop is placed on the rough surface such that it has an angular velocity $\omega = 4$ rad/s and an angular deceleration $\alpha = 5$ rad/s$^2$. Also, its centre has a velocity of $v_0 = 5$ m/s and a deceleration $a_0 = 2$ m/s$^2$. Determine the magnitude of acceleration of point $B$ at this instant.



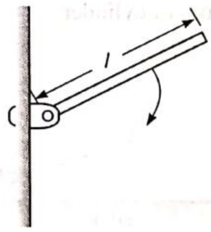

**37.** A boy of mass $m$ runs on ice with velocity $v_0$ and steps on the end of a plank of length $l$ and mass $M$ which is perpendicular to his path.




(a) Describe quantitatively the motion of the system after the boy is on the plank. Neglect friction with the ice.

(b) One point on the plank is at rest immediately after the collision. Where is it?

**38.** A thin plank of mass $M$ and length $l$ is pivoted at one end. The plank is released at 60° from the vertical. What is the magnitude and direction of the force on the pivot when the plank is horizontal?



# ANSWERS

## Introductory Exercise 9.1

**1.** About a diagonal, because the mass is more concentrated about a diagonal **2.** $\frac{\pi^2}{3}$ **3.** $\frac{2}{\sqrt{3}} l$

**4.** (i) $\frac{8}{5} mr^2 + 2ma^2$ (ii) $\frac{8}{5} mr^2 + ma^2$ **5.** (a) $2Ml^2$ (b) $\frac{1}{3} Ml^2$ **6.** $\frac{2}{3} Ma^2$

**7.** 0.5 kg-m² **8.** 0.43 kg-m² **9.** $\frac{R}{\sqrt{2}}$ **10.** The one having the smaller density

## Introductory Exercise 9.2

**1.** 100 rad **2.** 800 rad **3.** 5 N-m **4.** 0.87 N **5.** (a) 4 rad-s$^{-1}$, − 6 rad-s$^{-2}$ (b) − 12 rad-s$^{-2}$

**6.** 7 s

## Introductory Exercise 9.3

**2.** $\frac{1}{2} mRv$ **3.** $2\sqrt{2}\, mv$ **4.** $\frac{mu^3 \cos\alpha \sin^2\alpha}{2g}$ **5.** No

## Introductory Exercise 9.4

**1.** $\frac{\omega_0 M}{M + 2m}$ **2.** Duration of day-night incerease **3.** True

## Introductory Exercise 9.5

**1.** $y^2 = \left(\frac{2a_0}{\omega^2}\right) x$ **2.** $\sqrt{\frac{3g}{l}(1 - \sin\theta)}$

## Introductory Exercise 9.6

**1.** $\frac{2}{7} mgh$ **2.** $\pm \cos^{-1}\left(\frac{v}{R\omega}\right)$ **3.** $\frac{v_1 - v_2}{2R}$

## Introductory Exercise 9.7

**1.** (a) $g \sin\theta - \mu g \cos\theta$ (b) $\frac{5}{2} \frac{\mu g \cos\theta}{R}$ **2.** False **3.** Leftwards **4.** False **5.** $\frac{I + 2Mr^2}{4Mr^2 - I}$

**6.** $\lim_{F \to 0}$ can make the body move **7.** False

## Introductory Exercise 9.8

**1.** (a) $\mu < 1$ (b) $\mu > 1$ **2.** $\frac{2}{5} R$

# AIEEE Corner

## Subjective Questions (Level 1)

**1.** $\frac{5}{3} ml^2$ **2.** 55 kg-m² **3.** $\frac{l}{\sqrt{2}}$ **4.** $\frac{Ma^2}{12}$ **5.** 8 cm

**6.** $I = \mu r^2$, where $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is called the reduced mass of two masses.

**7.** $I = \left(\frac{\alpha l^4}{4} + \frac{\beta l^3}{3}\right)$ **8.** $\frac{\pi}{30}$ rad-s$^{-1}$ **9.** $(-\hat{k})$ rad-s$^{-1}$ **10.** 10 rad-s$^{-1}$ **11.** $\omega = \frac{v}{2R}$ **12.** 2 rad/s

**13.** $(-2\hat{i} - 2\hat{k})$ N·m **14.** 400 N·m (perpendicular to the plane of motion) **15.** 2.71 N·m **16.** $\frac{83}{20}$ N·m

**17.** $(8\pi)$ rad·s$^{-2}$, $(40\pi)$ rad·s$^{-1}$ **18.** 70 rad **19.** (a) 0.01 N·m (b) 0.13 N·m **20.** 20 s

**21.** $\frac{20}{3}$ N·m, 4 s **22.** (a) $\frac{3\omega_0 R}{4\mu g}$ (b) $\frac{3\omega_0^2 R}{8\mu g}$ **23.** (a) 36 s (b) $12\sqrt{\frac{2}{k}}$

**24.** (a) $\omega = 12.5$ rad·s$^{-1}$ (b) 127.5 rad **25.** 9 rad, 1.43 **26.** $\omega_{av} = 4$ rad/s, $\alpha_{av} = -6.0$ rad·s$^{-2}$

**27.** $4\sqrt{2}$ kg·m$^2$s$^{-1}$ **28.** $ml^2\omega$ **29.** $-\left(\frac{7}{5} mRv\right)\hat{k}$ **30.** $-\left(\frac{10}{3} mRv\right)\hat{k}$ **31.** Increase **32.** $\frac{25}{6}$ rad·s$^{-1}$

**33.** $7.29 \times 10^{-4}$ kg·m$^2$ **34.** (a) 14.3 rad·s$^{-1}$ = 2.27 rev·s$^{-1}$ (b) $E_i = 39.9$ J, $E_f = 181$ J (c) 141.1 J

**35.** (a) $\omega = \left(1 + \frac{2m}{M}\right)\omega_0$ (b) $\frac{1}{2} m\omega_0^2 R^2 \left(1 + \frac{2m}{M}\right)$ **36.** $\frac{4F\cos\theta}{3M+8m}, \frac{3MF\cos\theta}{3M+8m}, \frac{MF\cos\theta}{3M+8m}$ **37.** $\frac{F}{M+3m}$

**38.** 72 N **39.** (a) $\frac{2}{7\sqrt{3}}$ (b) $\frac{25}{7}$ ms$^{-2}$ (c) $\frac{30}{7}$ ms$^{-2}$ **40.** $\frac{4}{3} ml\omega$ **41.** (a) $\frac{2v}{9L}$ (b) $\frac{1}{9}$

**42.** (a) $\frac{2u}{3}$ (b) $\frac{2u}{\sqrt{3}}$

### Objective Questions (Level 1)

**1.** (d) **2.** (a) **3.** (c) **4.** (d) **5.** (b) **6.** (a) **7.** (c) **8.** (b) **9.** (b) **10.** (b)
**11.** (d) **12.** (a) **13.** (b) **14.** (b) **15.** (b) **16.** (d) **17.** (a) **18.** (b) **19.** (b) **20.** (d)
**21.** (c) **22.** (d) **23.** (b) **24.** (a) **25.** (a) **26.** (b) **27.** (c) **28.** (b) **29.** (a) **30.** (d)

## JEE Corner

### Assertion and Reason

**1.** (d) **2.** (b) **3.** (d) **4.** (a) **5.** (a) **6.** (a) **7.** (c) **8.** (b) **9.** (b) **10.** (a)
**11.** (c)

### Objective Questions (Level 2)

**1.** (d) **2.** (b) **3.** (b) **4.** (d) **5.** (c) **6.** (a) **7.** (a) **8.** (b) **9.** (c) **10.** (c)
**11.** (b) **12.** (c) **13.** (b) **14.** (b) **15.** (d) **16.** (a) **17.** (b) **18.** (b) **19.** (d) **20.** (b)
**21.** (c) **22.** (b) **23.** (c) **24.** (d) **25.** (c) **26.** (a) **27.** (b) **28.** (d) **29.** (a) **30.** (c)
**31.** (b) **32.** (b) **33.** (b) **34.** (c) **35.** (d) **36.** (a) **37.** (d) **38.** (b) **39.** (d) **40.** (c)
**41.** (a) **42.** (a) **43.** (a) **44.** (c) **45.** (a) **46.** (d) **47.** (b) **48.** (d) **49.** (c) **50.** (c)
**51.** (d) **52.** (d) **53.** (a)

### More than One Correct Options

**1.** (a,c) **2.** (all) **3.** (a,b,d) **4.** (a,b) **5.** (a,c,d) **6.** (all) **7.** (a,c)
**8.** (b,d) **9.** (a,b) **10.** (a,c) **11.** (a,b)

### Match the Columns

**1.** (a) → q (b) → s (c) → q (d) → p
**2.** (a) → p (b) → s (c) → p (d) → r
**3.** (a) → p,s (b) → p,r (c) → q,r (d) → p,r
**4.** (a) → q (b) → r (c) → r (d) → p
**5.** (a) → q (b) → s (c) → p (d) → s
**6.** (a) → q,r (b) → p (c) → s (d) → q
**7.** (a) → r (b) → q (c) → s (d) → p

## Subjective Questions (Level 2)

**1.** In each case in clockwise direction **2.** (a) $3g/2$, (b) $3g/4$ (c) $mg/4$ **3.** 0.612 m **4.** $\frac{F}{2M}$, $\frac{F}{2}$

**5.** 0.75 ms$^{-1}$, 1.98 ms$^{-1}$ **6.** (b) $\frac{1}{3}Mg$ **7.** $\sqrt{\frac{4mg}{(2m+M)R}}$ **8.** $\frac{2v_0}{R}$

**9.** $f = \frac{2}{7}mg\sin\theta$, $N = \frac{mg}{7}(17\cos\theta - 10) + \frac{mv_0^2}{(R-r)}$

**10.** (a) $f = \frac{2}{3}\left(\frac{1}{2} - \frac{r}{R}\right)F$ assuming $f$ opposite to $F$ (b) $a = \left(\frac{2F}{3mR}\right)(R+r)$ (c) yes, if $r$ is greater than $\frac{1}{2}R$.

(d) $f$ in same direction as $F$.

**11.** (a) $\frac{8F}{3m_1+8m_2}$, $\frac{4F}{3m_1+8m_2}$

(b) $\frac{3m_1F}{3m_1+8m_2}$ (between plank and cylinder) $\frac{m_1F}{3m_1+8m_2}$ (between cylinder and ground)

**12.** (a) 1.43 ms$^{-2}$ (b) 7.15 rad-s$^{-2}$ (c) 1.43 N **13.** 2.25 s **14.** 0.745 ms$^{-1}$ (rightwards)

**15.** $\frac{3}{2}\frac{v}{r}$ (anticlockwise) **16.** 1 m **17.** $\frac{\omega_0 R}{6}$; $\frac{\omega_0}{6}$ **18.** $S = \frac{2\omega_0^2 r^2}{81\mu g}$ **19.** $v = 1.5$ ms$^{-1}$ **20.** $\frac{9}{2}\frac{mv_0^2}{l}$

**21.** (a) Sphere, 5.29 ms$^{-1}$, disk 5.11 ms$^{-1}$, hoop 4.43 ms$^{-1}$ (b) Sphere 4.2 N, disk 4.9 N, hoop 7.36 N

(c) Sphere, 1.51 s disk 1.56 s hoop 1.81 s

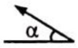
**22.** $2l\sqrt{\frac{g\sqrt{3}}{l}}$ **24.** (a) $\left(\frac{8}{3}m\right)\sqrt{3gl}$, (b) $\frac{4}{3}m\sqrt{6gl}\,(\sqrt{2}+1)$

**25.** (a) $\frac{4}{7}mg\sqrt{1+\left(\frac{7l\omega^2}{4g}\right)^2}$ (b) $\sqrt{\left(\frac{3g}{7}\right)^2 + (l\omega^2)^2}$ (c) $\left(\frac{13}{7}mg + ml\omega^2\right), \left(\frac{6g}{7} + l\omega^2\right)$ (d) $\sqrt{\frac{6g}{7l}}$

**26.** $\frac{Ml^2}{12d^2 + l^2}$ **27.** $\omega = \frac{6v_0}{l}\left(\frac{\cos\theta}{1+3\cos^2\theta}\right)$ **28.** (a) $\sqrt{3}\,ml\omega^2$ (b) $F_x = -\frac{F}{4}$, $F_y = \sqrt{3}\,ml\omega^2$

**29.** 2.0 ms$^{-1}$, 1.5 ms$^{-1}$, 16.67 N **30.** $d_1 = \frac{r^2\omega_0^2(\mu\cos\alpha - \sin\alpha)}{2g(3\mu\cos\alpha - \sin\alpha)^2}$, $d_{max} = \frac{r^2\omega_0^2(\mu\cos\alpha - \sin\alpha)}{4g\sin\alpha\,(3\mu\cos\alpha - \sin\alpha)}$

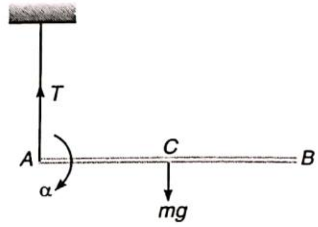
**32.** $\theta = \tan^{-1}\left(\frac{4\mu}{13}\right)$ **33.** 3.6 ms$^{-2}$, $\frac{4gt}{11}$ **34.** $\frac{F}{26M}$, $\frac{21F}{26M}$ **35.** $\sqrt{\frac{6g\sqrt{3}}{11r}}$, $\frac{3g}{11r}$ **36.** 6.21 ms$^{-2}$

**37.** (b) $\frac{2l}{3}$ from the boy **38.** $\frac{\sqrt{10}}{4}Mg$,  $\alpha = \tan^{-1}\left(\frac{1}{3}\right)$

# Chapter 9 Mechanics of Rotational Motion

## Level 2

1. Torque about bottommost point in each case is clockwise.




2. $$\alpha = \frac{\tau}{I} = \frac{mg\left(\frac{l}{2}\right)}{\frac{ml^2}{3}} = \frac{3}{2}\frac{g}{l}$$

(a) $$a_B = l\alpha = \frac{3g}{2}$$ **Ans.**

(b) $$a_C = \frac{l}{2}\alpha = \frac{3g}{4}$$ **Ans.**

(c) $$a_C = \frac{mg - T}{m} \quad \text{or} \quad \frac{3g}{4} = \frac{mg - T}{m} \quad \therefore \quad T = \frac{mg}{4}$$ **Ans.**

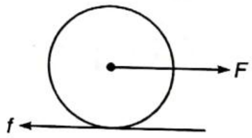
3. $$mgh = K_R + K_T = \frac{3}{4}mv^2$$

Here $h = s\sin\theta$

$\therefore$ $$gs\sin\theta = \frac{3}{4}v^2$$

or $$s = \frac{3v^2}{4g\sin\theta} = \frac{3 \times (2.0)^2}{4 \times 9.8 \times \frac{1}{2}} = 0.612 \text{ m}$$ **Ans.**

4. $I = MR^2$




For pure rolling to take place. $$a = R\alpha$$

or $$\frac{F - f}{M} = R\left(\frac{f.R}{MR^2}\right) = \frac{f}{M}$$

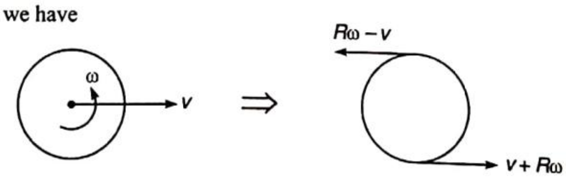
$\therefore$ $$f = \frac{F}{2}$$ **Ans.**

and $$a = \frac{F - f}{M} = \frac{F}{2M}$$ **Ans.**

5. $$v + R\omega = 1.5 \quad \text{...(i)}$$

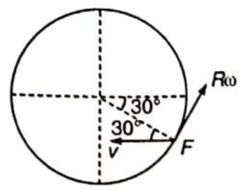
and $$R\omega - v = 3.0 \quad \text{...(ii)}$$

From Eq. (i) and (ii), we have




$$R\omega = 2.25 \text{ m/s} \quad \text{and} \quad v = -0.75 \text{ m/s}$$

Thus, velocity of point $C$ is 0.75 m/s (towards left).




$$v_F = \sqrt{v^2 + (R\omega)^2 + 2v(R\omega)\cos(90 + 30)}$$

$$= \sqrt{0.5625 + 5.0625 + 2 \times 0.75 \times 2.25 \times \left(-\frac{1}{2}\right)}$$

$= 1.98$ m/s **Ans.**

**6.** $$a = \frac{Mg - T}{M} \quad \text{...(i)}$$

$$\alpha = \frac{TR}{\frac{1}{2}MR^2} = \frac{2T}{MR} \quad \text{...(ii)}$$

$$a = R\alpha \quad \text{...(iii)}$$

Solving these three equations, we get

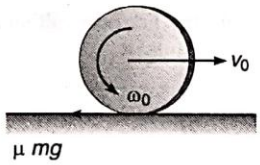
$$a = \frac{2g}{3} \text{ and } T = \frac{Mg}{3}$$ **Ans.**

**7.** From conservation of mechanical energy, decrease in gravitational PE = increase in rotational KE

or $$mg(R) = \left[\frac{1}{2}MR^2 + mR^2\right]\left(\frac{1}{2}\omega^2\right)$$

or $$\omega = \sqrt{\frac{4mg}{(2m + M)R}}$$ **Ans.**

**8.** Initially there is forward slipping. Therefore, friction is backwards and maximum. Let velocity becomes zero in time $t_1$ and angular velocity becomes zero in time $t_2$.




Then, $$0 = v_0 - at_1$$

or $$t_1 = \frac{v_0}{a} = \frac{v_0}{\mu g} \quad \text{...(i)}$$

and $$0 = \omega_0 - \alpha t_2 \quad \text{or} \quad t_2 = \frac{\omega_0}{\alpha}$$

Here, $$\alpha = \frac{\mu mgR}{\frac{1}{2}mR^2} = \frac{2\mu g}{R}$$

$\therefore$ $$t_2 = \frac{\omega_0 R}{2\mu g} \quad \text{...(ii)}$$

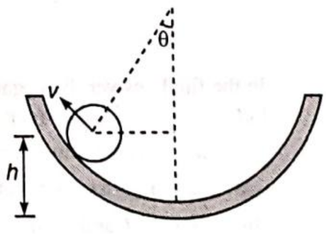
Disk will return back when $$t_2 > t_1$$

or $$\frac{\omega_0 R}{2\mu g} > \frac{v_0}{\mu g}$$

or $$\omega_0 > \frac{2v_0}{R}$$ **Ans.**

**9.** $$h = (R - r)(1 - \cos\theta) \quad \text{...(i)}$$

Kinetic energy at angle $\theta$ is, $K = \frac{7}{5}\left(\frac{1}{2}mv_0^2\right) - mgh$




$\because$ In case of pure rolling $\quad K_T = \frac{5}{7}K$

$\therefore$ $$\frac{1}{2}mv^2 = \frac{1}{2}mv_0^2 - \frac{5}{7}mgh$$

$\therefore$ $$v^2 = v_0^2 - \frac{10}{7}gh \quad \text{...(ii)}$$

Equation of motion at angle θ is,

$$N - mg\cos\theta = \frac{mv^2}{(R-r)}$$

$$\therefore \qquad N = mg\cos\theta + \frac{m}{(R-r)}\left(v_0^2 - \frac{10}{7}gh\right)$$

Substituting value of $h$ from Eq. (i)

$$N = mg\cos\theta + \left(\frac{m}{R-r}\right)\left\{v_0^2 - \frac{10}{7}g(R-r)(1-\cos\theta)\right\}$$

$$= \frac{mg}{7}(17\cos\theta - 10) + \frac{mv_0^2}{(R-r)} \qquad \textbf{Ans.}$$

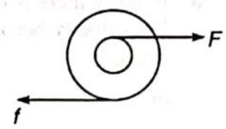
Force of friction, $$f = \frac{mg\sin\theta}{1+\frac{mr^2}{I}} \qquad \text{(for pure rolling to take place)}$$

$$= \frac{mg\sin\theta}{1+\frac{5}{2}} \qquad \left(I = \frac{2}{5}mr^2\right)$$

$$= \frac{2}{7}mg\sin\theta \qquad \textbf{Ans.}$$

**10.** (a) For pure rolling to take place,

$$a = R\alpha$$




or $$\frac{F-f}{m} = R\left[\frac{Fr + fR}{\frac{1}{2}mR^2}\right]$$

Solving this equation, we get

$$f = \frac{2}{3}\left(\frac{1}{2} - \frac{r}{R}\right)F \qquad \textbf{Ans.}$$

(b) Acceleration $$a = \frac{F-f}{m}$$

Substituting value of $f$ from part (a), we get

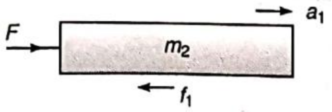
$$a = \frac{2F}{3mR}(R+r) \qquad \textbf{Ans.}$$

(c) $a > \frac{F}{m}$ if

$$\frac{2}{3R}(R+r) > 1 \qquad \text{or} \qquad r > \frac{R}{2} \qquad \textbf{Ans.}$$

(d) In this case force of friction is in forward direction. **Ans.**

**11.** We can choose any arbitrary directions of frictional forces at different contacts.




In the final answer the negative value will show the opposite directions.

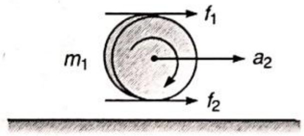
Let $f_1$ = friction between plank and cylinder $\qquad$ $f_2$ = friction between cylinder and ground

$a_1$ = acceleration of plank $\qquad$ $a_2$ = acceleration of centre of mass of cylinder

and $\alpha$ = angular acceleration of cylinder about is COM.

Directions of $f_1$ and $f_2$ are as shown here :

Since, there is no slipping anywhere




$$\therefore \qquad a_1 = 2a_2 \qquad \text{...(i)}$$

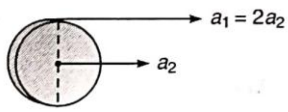
(Acceleration of plank = acceleration of top point of cylinder)

$$a_1 = \frac{F - f_1}{m_2} \qquad \text{...(ii)}$$

$$a_2 = \frac{f_1 + f_2}{m_1} \qquad \text{...(iii)}$$

$$\alpha = \frac{(f_1 - f_2)R}{I}$$

($I$ = moment of inertia of cylinder about COM)



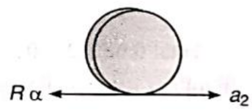

$$\therefore \qquad \alpha = \frac{(f_1 - f_2)R}{\frac{1}{2}m_1 R^2}$$

$$\alpha = \frac{2(f_1 - f_2)}{m_1 R} \qquad \text{...(iv)}$$

$$a_2 = R\alpha = \frac{2(f_1 - f_2)}{m_1} \qquad \text{...(v)}$$

(Acceleration of bottommost point of cylinder = 0)

(a) Solving Eqs. (i), (ii), (iii) and (v), we get




$$a_1 = \frac{8F}{3m_1 + 8m_2} \quad \text{and} \quad a_2 = \frac{4F}{3m_1 + 8m_2}$$

(b)
$$f_1 = \frac{3m_1F}{3m_1 + 8m_2}$$

$$f_2 = \frac{m_1F}{3m_1 + 8m_2}$$

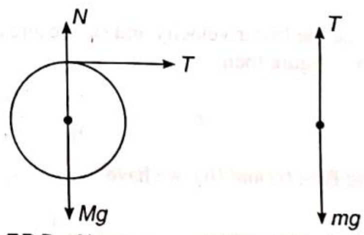
Since, all quantities are positive, they are correctly shown in figures.

**Note** *Above calculations have been done at $t = 0$ when $\omega = 0$.*

**12.** If $\alpha$ be the angular acceleration of the hoop and $a$ be the acceleration of its centre, acceleration of $m$ would be $\alpha r + a$.
Here, $Tr = I\alpha$ [where $I$ = moment of inertia of the hoop about the horizontal axis passing through its centre]

Also, $T = Ma$ and $mg - T = m[a + \alpha r]$

Solving, we get




F.B.D. of hoop F.B.D. of block of mass $m$

$$a = \frac{mg}{[M + 2m]} = \frac{2}{1.4} = 1.43\ \text{m/s}^2$$

Hence, $T = 1.43$ N

and $\alpha = \dfrac{Tr}{I} = \dfrac{T}{Mr} = 7.15\ \text{rad/s}^2$

**13.** Initially the cylinder will slip on the plank, therefore kinetic friction will act between the cylinder and the plank.

$$a_c = -\frac{\mu mg}{m} = -\mu g$$

$$a_p = +\frac{\mu mg}{2m} = +\frac{\mu g}{2}$$

$$\alpha_c = +\frac{(\mu mg)(R)}{(mR^2/2)} = +\frac{2\mu g}{R}$$

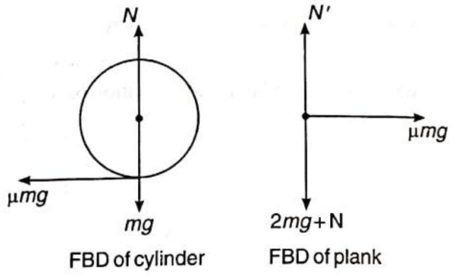
For pure rolling,

$$v_p = v_c - R\omega_c$$

$$\therefore \quad \frac{\mu g}{2} t = v_0 - \mu g t - (R)\left(\frac{2\mu g}{R}\right)(t)$$

$$\therefore \quad t = \frac{v_0}{3.5\mu g} = \frac{7}{3.5 \times 0.1 \times 10} = 2 \text{ s}$$

$$\therefore \quad s_c - s_p = v_0 t - \frac{1}{2} \times (\mu g)(t^2) - \frac{1}{2}\left(\frac{\mu g}{2}\right)(t^2)$$




FBD of cylinder FBD of plank

$$= (7 \times 2) - \frac{1}{2}(0.1)(10)(4) - \frac{1}{2}\left(\frac{0.1 \times 10}{2}\right)(4) = 11 \text{ m}$$

Also,

$$v_c - v_P = (v_0 - \mu g t) - \left(\frac{\mu g}{2}\right)(t)$$

$$= 7 - 0.1 \times 10 \times 2 - \frac{0.1 \times 10 \times 2}{2} = 4 \text{ m/s}$$

Hence, the remaining distance $(12 - 11 = 1 \text{ m})$ is travelled in a time,

$$t' = \frac{1.0}{4} = 0.25 \text{ s}$$

$\therefore$ Total time = 2 + 0.25 = 2.25 s

**14.** In rolling without sliding on a stationary ground, work done by friction is zero. Hence

work done by the applied force = change in kinetic energy

$$\therefore \quad (30)(0.25) = \frac{1}{2} \times 9 \times v^2 + 2\left[\frac{1}{2} \times 6 \times v^2 + \frac{1}{2} \times \frac{1}{2} \times 6 \times r^2 \times \frac{v^2}{r^2}\right]$$

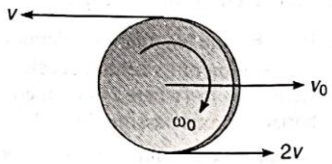
or

$$7.5 = 13.5v^2$$

$$\therefore \quad v = 0.745 \text{ m/s}$$

**Ans.**

**15.** Let $v_0$ be the linear velocity and $\omega_0$ the angular velocity of the disc as shown in figure then,




$$v_0 - r\omega_0 = 2v \quad \ldots\text{(i)}$$

and

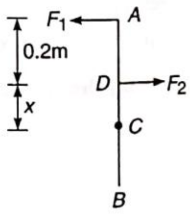
$$v_0 + r\omega_0 = -v \quad \ldots\text{(ii)}$$

Solving Eqs. (i) and (ii), we have $\omega_0 = -\frac{3}{2}\frac{v}{r}$

Hence, the angular velocity of disc is $\frac{3}{2}\frac{v}{r}$ anticlockwise.

**Ans.**

**16.** Let $x$ be the distance of centre point $C$ of rod from $D$. Then,




$$F_2 - F_1 = ma \quad \text{or} \quad F_1 = 3 \text{ N}$$

Further,

$$\tau_c = 0$$

$$\therefore \quad F_2 x = F_1(0.2 + x)$$

$$5x = F_1(0.2 + x)$$

$$\therefore \quad 5x = 3(0.2 + x)$$

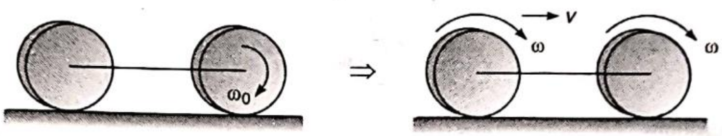
or

$$x = 0.3 \text{ m}$$

$$\therefore \quad \text{Length of rod} = 2(x + 0.2) = 1.0 \text{ m}$$

**Ans.**

**17.** $L_i = L_f$ (about bottommost point)




$\therefore$ $$I\omega_0 = 2[I\omega + mRv]$$

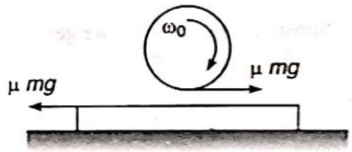
or $$\left(\frac{1}{2}mR^2\right)\omega_0 = 2\left[\frac{1}{2}mR^2\omega + mR(\omega R)\right]$$

$\therefore$ $$\omega = \frac{\omega_0}{6}$$ **Ans.**

and $$v = \omega R = \frac{\omega_0 R}{6}$$ **Ans.**

**18.** Let, $a_1$ = linear acceleration of sphere (towards right), $a_2$ = linear acceleration of plank (towards left)

and $\alpha$ = angular retardation of sphere




$$a_1 = a_2 = \frac{\mu mg}{m} = \mu g$$

$$\alpha = \frac{\mu mgr}{\frac{2}{5}mr^2} = \frac{5}{2}\frac{\mu g}{r}$$

Let pure rolling starts after time '$t$'. Then $\omega r - v = v$

$\therefore$ $$\omega r = 2v$$

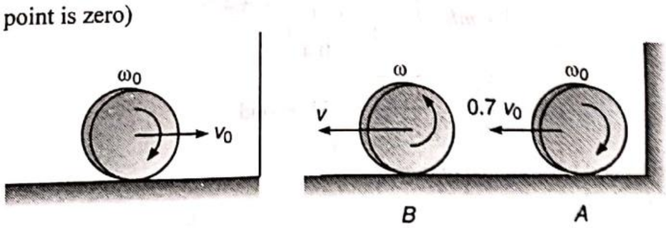
$$(\omega_0 - \alpha t)r = 2(a_1 t)$$

Substituting the values, $$t = \frac{2}{9}\frac{\omega_0 r}{\mu g}$$

$\therefore$ $$s = \frac{1}{2}(a_2)t^2 = \frac{2\omega_0^2 r^2}{81\mu g}$$ **Ans.**

**19.** Between $A$ and $B$, there is forward slipping. Therefore, friction will be maximum and backwards (rightwards). At point $B$ where $v = R\omega$, ball starts rolling without slipping and force of friction becomes zero.

From conservation of angular momentum between points $A$ and $B$ about bottommost point (because torque of friction about this point is zero)




$$L_A = L_B$$

$\therefore$ $$m(0.7v_0)R - I\omega_0 = mvR + I\omega$$

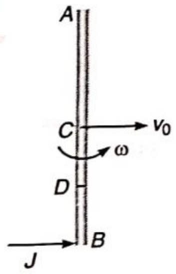
Substituting $\omega_0 = \frac{v_0}{R}$, $\omega = \frac{v}{R}$ and $I = \frac{2}{5}mR^2$, we get

$$v = \frac{3}{14}v_0 = \left(\frac{3}{14}\right)(7)\text{ m/s} = 1.5\text{ m/s}$$ **Ans.**

**20.** Let $J$ be the linear impulse applied at $B$ and $\omega$ the angular speed of rod.

$$J = mv_0 \qquad \text{...(i)}$$

$$J\left(\frac{l}{2}\right) = \frac{ml^2}{12} \cdot \omega \qquad \text{...(ii)}$$




Solving these two equations, $\omega = \frac{6v_0}{l}$

Linear speed of $D$ (mid-point of $CB$) relative to $C$,

$$v = \omega\left(\frac{l}{4}\right) = \frac{3}{2} v_0$$

$\therefore$ Force exerted by upper half on the lower half,

$$F = \frac{\left(\frac{m}{2}\right) v^2}{\left(\frac{l}{4}\right)}$$

Substituting $v = \frac{3}{2} v_0$, we get $F = \frac{9}{2} \frac{mv_0^2}{l}$ **Ans.**

**21.** $\frac{I}{mR^2} = \frac{2}{5} = 0.4$ for sphere

$= \frac{1}{2} = 0.5$ for disc and $= 1$ for hoop

$$s = \frac{2}{\sin 30°} = 4 \text{ m}$$

**For sphere :**

$$a = \frac{g \sin\theta}{1 + \frac{I}{mR^2}} = \frac{9.8 \times \frac{1}{2}}{1 + 0.4} = 3.5 \text{ m/s}$$

$\therefore$ $v = \sqrt{2as} = \sqrt{2 \times 3.5 \times 4} = 5.29$ m/s

$$f = \frac{mg\sin\theta}{1 + mR^2/I} = \frac{3 \times 9.8 \times \frac{1}{2}}{1 + \left(\frac{1}{0.4}\right)} = 4.2 \text{ N}$$

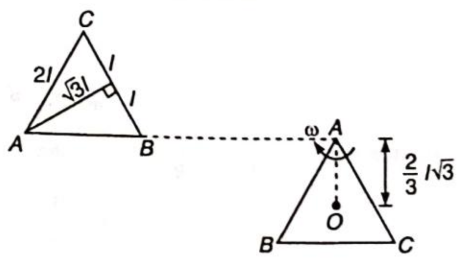
$$t = \sqrt{\frac{2s}{a}} = \sqrt{\frac{2 \times 4}{3.5}} = 1.51 \text{ second}$$

Similarly the values for disk and hoop can be obtained.

**22.** $I_A = I_{AB} + I_{AC} + I_{BC}$

$= \frac{4}{3} ml^2 + \frac{4}{3} ml^2 + \left\{\frac{1}{3} ml^2 + m(l\sqrt{3})^2\right\}$

$= 6\,ml^2$



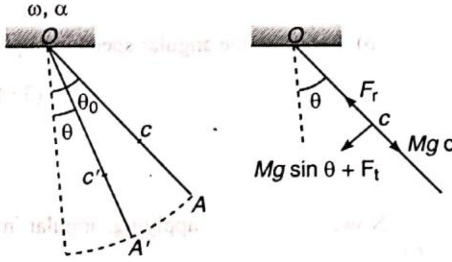

If $\omega$ is the angular velocity in the second position, then using conservation of mechanical energy, we have

**of COM :** $h_i = +\dfrac{\sqrt{3}l}{3}$ and $h_f = -\dfrac{2\sqrt{3}}{3}l$

$$3mg\left(\frac{l\sqrt{3}}{3}\right) = \frac{1}{2}(6ml^2)\omega^2 + 3mg\left(\frac{-2l\sqrt{3}}{3}\right) \quad \text{or} \quad \omega = \sqrt{\frac{g\sqrt{3}}{l}}$$

Now, velocity of $C$ at this instant is $2l\omega$ or $2\sqrt{gl\sqrt{3}}$ and maximum. **Ans.**

**23.** (i) $C$ is the centre of mass of the rod. Let $\omega$ be the angular speed of rod about point $O$ at angle $\theta$. From conservation of mechanical energy,




$$Mg\frac{L}{2}(\cos\theta - \cos\theta_0) = \frac{1}{2}\left(\frac{ML^2}{3}\right)\omega^2$$

$\therefore$ $$\omega^2 = \frac{3g}{L}(\cos\theta - \cos\theta_0) \qquad \text{...(i)}$$

Now, $$F_r - Mg\cos\theta = M\left(\frac{L}{2}\right)\omega^2 \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$F_r = \frac{1}{2}Mg(5\cos\theta - 3\cos\theta_0)$$ **Hence proved.**

(ii) Angular acceleration of rod at this instant,

$$\alpha = \frac{\tau}{I} = \frac{Mg\frac{L}{2}\sin\theta}{\frac{ML^2}{3}} = \frac{3}{2}\frac{g\sin\theta}{L}$$

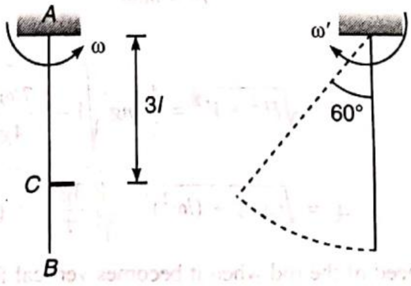
Tangential acceleration of COM, $$a_t = (\alpha)\left(\frac{L}{2}\right) = \frac{3}{4}g\sin\theta \qquad \text{...(iii)}$$

Now, $$F_t + Mg\sin\theta = Ma_t \qquad \text{...(iv)}$$

From Eqs. (iii) and (iv), we get $$F_t = -\frac{1}{4}Mg\sin\theta$$ **Hence proved.**

Here negative sign implies that direction of $F_t$ is opposite to the component $Mg\sin\theta$.

**24.** (a) From conservation of mechanical energy.




$$(3m)(g)(2l) = \frac{1}{2}I\omega^2 = \frac{1}{2}\left[\frac{(3m)(4l)^2}{3}\right]\omega^2 = 8ml^2\omega^2$$

$\therefore$ $$\omega = \frac{1}{2}\sqrt{\frac{3g}{l}}$$

Applying, angular impulse = change in angular momentum

$$J(3l) = I\omega$$

or $$3Jl = (16ml^2)\left(\frac{1}{2}\sqrt{\frac{3g}{l}}\right)$$

$\therefore$ $$J = \frac{8}{3}ml\sqrt{\frac{3g}{l}}$$

or $$J = \frac{8}{3}m\sqrt{3gl}$$ **Ans.**

(b) Let $\omega'$ be the angular speed in opposite direction. Again applying conservation of mechanical energy,

$$(3m)(g)(l) = \frac{1}{2}I(\omega')^2 = 8ml^2(\omega')^2$$

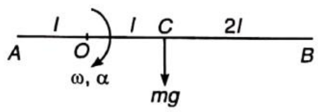
$\therefore$ $$\omega' = \frac{1}{2\sqrt{2}}\sqrt{\frac{3g}{l}}$$

Now, applying, angular impulse = change in angular momentum

$\therefore$ $$J(3l) = I(\omega + \omega') = (16ml^2)\frac{1}{2}\sqrt{\frac{3g}{l}}\left(1 + \frac{1}{\sqrt{2}}\right)$$

$\therefore$ $$J = \frac{4}{3}m\sqrt{6gl}\,(\sqrt{2} + 1)$$ **Ans.**

**25.** $$\alpha = \frac{mgl}{\frac{m(4l)^2}{12} + ml^2} = \frac{3}{7}\cdot\frac{g}{l}$$




$\therefore$ $$(a_C)_V = l\alpha = \frac{3}{7}g \quad \text{(downwards)}$$

Let $V$ be the vertical reaction (upwards) at axis, then

$$mg - V = ma_C = \frac{3mg}{7}$$

$\therefore$ $$V = \frac{4}{7}mg \quad \text{...(i)}$$

If $H$ be the horizontal reaction (towards $CO$) at axis, then

$$H = ml\omega^2 \quad \text{...(ii)}$$

$\therefore$ Total reaction at axis,

$$N = \sqrt{H^2 + V^2} = \frac{4}{7}mg\sqrt{1 + \left(\frac{7l\omega^2}{4g}\right)^2}$$ **Ans.**

(b) $$a_C = \sqrt{(a_C)_V^2 + (l\omega^2)^2} = \sqrt{\left(\frac{3g}{7}\right)^2 + (l\omega^2)^2}$$ **Ans.**

(c) Let $\omega'$ be the angular speed of the rod when it becomes vertical for the first time. Then from conservation of mechanical energy,

$$\frac{1}{2}I(\omega'^2 - \omega^2) = mgl$$

$\therefore$ $$\omega'^2 = \omega^2 + \frac{2mgl}{I}$$

$$= \omega^2 + \frac{2mgl}{\frac{7}{3}ml^2} = \omega^2 + \frac{6}{7}\frac{g}{l}$$

Acceleration of centre of mass at this instance will be,

$$a_C = l\omega'^2 = l\omega^2 + \frac{6g}{7}$$ **Ans.**

Let $V$ be the reaction (upwards) at axis at this instant, then,

$$V - mg = ma_C = ml\omega^2 + \frac{6mg}{7}$$

$\therefore$ $$V = \frac{13}{7}mg + ml\omega^2$$ **Ans.**

(d) From conservation of mechanical energy,

$$mgl = \frac{1}{2}I\omega_{min}^2$$

$\therefore$ $$\omega_{min} = \sqrt{\frac{2mgl}{I}} = \sqrt{\frac{2mgl}{\frac{7}{3}ml^2}} = \sqrt{\frac{6}{7}\frac{g}{l}}$$ **Ans.**

**26.** Linear momentum, angular momentum and kinetic energy are conserved in the process.
From conservation of linear momentum,

$$Mv' = mv$$

or $$v' = \frac{m}{M}v \quad \text{...(i)}$$

Conservation of angular momentum gives,

$$mvd = \left(\frac{Ml^2}{12}\right)\omega$$

or $$\omega = \left(\frac{12mvd}{Ml^2}\right) \quad \text{...(ii)}$$

Collision is elastic. Hence,

$$e = 1$$

or relative speed of approach = relative speed of separation

$\therefore$ $$v = v' + d\omega$$

Substituting the values, we have $$v = \frac{m}{M}v + \frac{12mvd^2}{Ml^2}$$

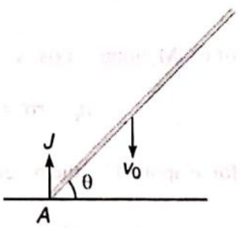
Solving it, we get $$m = \frac{Ml^2}{12d^2 + l^2}$$ **Ans.**

**27.** Let $v$ = linear velocity of rod after impact (upwards),
$\omega$ = angular velocity of rod
and $J$ = linear impulse at $A$ during impact

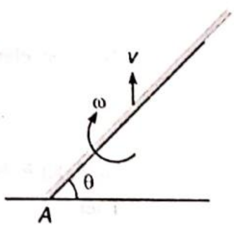
Then, $$J = \Delta P = P_f - P_i$$

$$J = mv - (-mv_0)$$

$\therefore$ $$J = m(v + v_0) \quad \text{...(i)}$$



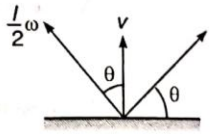
At the time of impact



After impact

Angular impulse = $\Delta L$

$\therefore$ $$J\left(\frac{l}{2}\cos\theta\right) = I\omega = \frac{ml^2}{12}\omega \quad \text{...(ii)}$$




Collision is elastic ($e = 1$)

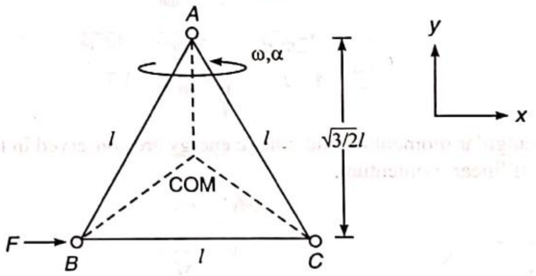
$\therefore$ Relative speed of approach = Relative speed of separation at point of impact

$$v_0 = v + \frac{l}{2}\omega\cos\theta \quad \text{...(iii)}$$

Solving above equations, we get $$\omega = \frac{6v_0\cos\theta}{l(1 + 3\cos^2\theta)}$$ **Ans.**

**28.** (a) The distance of centre of mass (COM) of the system about point $A$ will be :

$$r = \frac{l}{\sqrt{3}}$$




Therefore, the magnitude of horizontal force exerted by the hinge on the body is

$F$ = centripetal force

or $$F = (3m)r\omega^2$$

or $$F = (3m)\left(\frac{l}{\sqrt{3}}\right)\omega^2$$

or $$F = \sqrt{3}\,ml\omega^2$$

(b) Angular acceleration of system about point $A$ is

$$\alpha = \frac{\tau_A}{I_A}$$

$$= \frac{(F)\left(\frac{\sqrt{3}}{2}l\right)}{2ml^2}$$

$$= \frac{\sqrt{3}F}{4ml}$$

Now, acceleration of COM along $x$-axis is

$$a_x = r\alpha = \left(\frac{l}{\sqrt{3}}\right)\left(\frac{\sqrt{3}F}{4ml}\right) \text{ or } a_x = \frac{F}{4m}$$

Now, let $F_x$ be the force applied by the hinge along $x$-axis.

Then, $$F_x + F = (3m)a_x$$

or $$F_x + F = (3m)\left(\frac{F}{4m}\right)$$

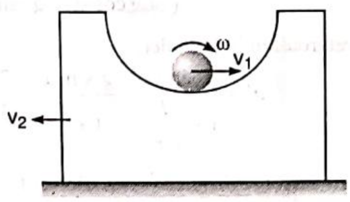
or $$F_x + F = \frac{3}{4}F \text{ or } F_x = -\frac{F}{4}$$

Further if $F_y$ be the force applied by the hinge along $y$-axis. Then,

$$F_y = \text{centripetal force}$$

or $$F_y = \sqrt{3}\, ml\omega^2$$

**29.** From conservation of linear momentum




$$mv_1 = Mv_2 \quad \text{...(i)}$$

∴ Velocity of cylinder axis relative to block $v_r = v_1 + v_2$ ...(ii)

Applying conservation of mechanical energy,

$$mg(R - r) = \frac{1}{2}mv_1^2 + \frac{1}{2}I\omega^2 + \frac{1}{2}Mv_2^2 \quad \text{...(iii)}$$

Here, $I = \frac{1}{2}mr^2$ and $\omega = \frac{v_r}{r}$

Solving the above equations with given data, we get

$v_1 = 2.0$ m/s and $v_2 = 1.5$ m/s **Ans.**

Further, $$N - mg = \frac{mv_r^2}{R - r}$$

∴ $$N = mg + \frac{mv_r^2}{R - r} = (0.5)(10) + \frac{(0.5)(3.5)^2}{0.525} = 16.67 \text{ N}$$ **Ans.**

**30.** Given $\mu > \tan\alpha \Rightarrow \mu mg\cos\alpha > mg\sin\alpha$


$$a = (\mu g\cos\alpha - g\sin\alpha)$$

$$\alpha = \frac{(\mu mg\cos\alpha)\,r}{\frac{1}{2}mr^2} = \frac{2\mu g\cos\alpha}{r}$$

Slipping will stop when,

$$v = r\omega$$

or $$at = r(\omega_0 - \alpha t)$$

$\therefore$ $$t = \frac{r\omega_0}{a + r\alpha} = \left(\frac{r\omega_0}{3\mu g \cos\alpha - g\sin\alpha}\right)$$

$$d_1 = \frac{1}{2}at^2 = \frac{1}{2}(\mu g\cos\alpha - g\sin\alpha)\left(\frac{r\omega_0}{3\mu g\cos\alpha - g\sin\alpha}\right)^2$$

$$= \frac{r^2\omega_0^2(\mu\cos\alpha - \sin\alpha)}{2g(3\mu\cos\alpha - \sin\alpha)^2}$$ **Ans.**

$$v = at = (\mu g\cos\alpha - g\sin\alpha)\left(\frac{r\omega_0}{3\mu g\cos\alpha - g\sin\alpha}\right) = \frac{r\omega_0(\mu\cos\alpha - \sin\alpha)}{(3\mu\cos\alpha - \sin\alpha)}$$

Once slipping is stopped, retardation in cylinder,

$$a' = \frac{g\sin\alpha}{1 + \frac{I}{mr^2}} = \frac{g\sin\alpha}{1 + \frac{1}{2}} = \frac{2}{3}g\sin\alpha$$

$$d_2 = \frac{v^2}{2a'} = \frac{3r^2\omega_0^2(\mu\cos\alpha - \sin\alpha)^2}{(3\mu\cos\alpha - \sin\alpha)^2(4g\sin\alpha)}$$

$\therefore$ $$d_{\max} = d_1 + d_2$$

$$= \frac{r^2\omega_0^2(\mu\cos\alpha - \sin\alpha)}{2g(3\mu\cos\alpha - \sin\alpha)^2}\left[1 + \frac{3(\mu\cos\alpha - \sin\alpha)}{2\sin\alpha}\right]$$

$$= \frac{r^2\omega_0^2(\mu\cos\alpha - \sin\alpha)}{4g\sin\alpha(3\mu\cos\alpha - \sin\alpha)}$$ **Ans.**

**Note** *Once slipping was stopped, pure rolling continues if*

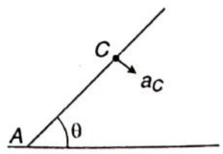
$$\mu > \frac{\tan\alpha}{1 + \frac{mr^2}{I}}$$

or $$\mu > \frac{\tan\alpha}{1 + 2} \quad \text{or} \quad \mu > \frac{\tan\alpha}{3}$$

and already in the question it is given that $\mu > \tan\alpha$. That's why we have taken $a' = \frac{2}{3}g\sin\alpha$.

**31.** Point $A$ is momentarily at rest.

$$\alpha = \frac{mg\frac{l}{2}\cos\theta}{\frac{ml^2}{3}} = \frac{3}{2}\frac{g\cos\theta}{l}$$




$\therefore$ $$a_C = \frac{l}{2}\alpha = \frac{3}{4}g\cos\theta$$

Now $$\mu N = ma_x \quad \text{or} \quad \mu N = ma_C\sin\theta$$

or $$\mu N = \frac{3}{4}mg\sin\theta\cos\theta \qquad \text{...(i)}$$

Further, $$mg - N = ma_y$$

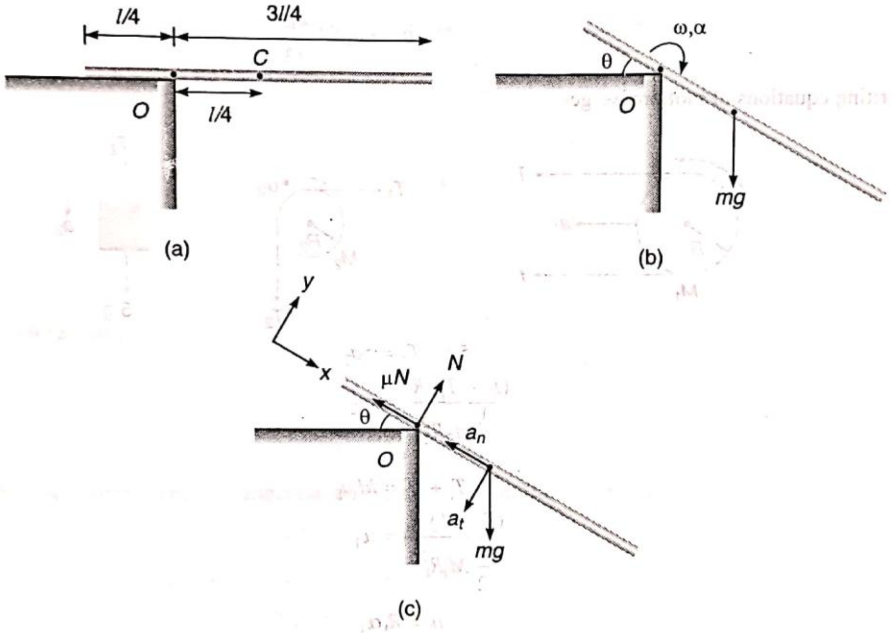
or $$N = mg - ma_C\cos\theta$$

or $$N = mg - \frac{3}{4}mg\cos^2\theta \qquad \text{...(ii)}$$

Dividing Eq. (i) by (ii), we have $$\mu = \frac{\frac{3}{4}\sin\theta\cos\theta}{1-\frac{3}{4}\cos^2\theta} = \frac{3\sin\theta\cos\theta}{4-3\cos^2\theta}$$

$$= \frac{3\sin\theta\cos\theta}{1+3\sin^2\theta}$$ **Proved.**

**32. Figure (a) and (b) :**




ω : Decrease in gravitational potential energy = increase in rotational kinetic energy

$\therefore$ $$mg\frac{l}{4}\sin\theta = \frac{1}{2}I_0\omega^2 = \frac{1}{2}\left[\frac{ml^2}{12} + m\left(\frac{l}{4}\right)^2\right]\omega^2$$

$\therefore$ $$\omega = \sqrt{\left[\frac{24g\sin\theta}{7l}\right]} \quad \text{...(i)}$$

α : $$\alpha = \frac{\tau}{I} = \frac{mg\frac{l}{4}\cos\theta}{\left[\frac{ml^2}{12} + m\left(\frac{l}{4}\right)^2\right]} = \frac{12\,g\cos\theta}{7l} \quad \text{...(ii)}$$

$$\Sigma F_y = ma_y \text{ or } mg\cos\theta - N = ma_t$$

or $$N = mg\cos\theta - ma_t = mg\cos\theta - m\frac{l}{4}\alpha$$

Substituting value of α from Eq. (ii), we get

$$N = \frac{4}{7}mg\cos\theta \quad \text{...(iii)}$$

Rod begins to slip when :

$$\mu N - mg \sin\theta = ma_n$$

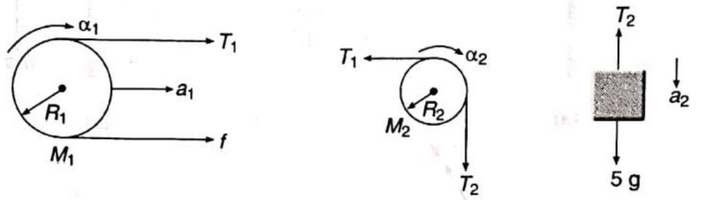
or
$$\frac{4}{7}\mu mg \cos\theta - mg\sin\theta = m\frac{l}{4}\omega^2$$

Substitution value of $\omega$ from Eq. (i), we get

$$\tan\theta = \frac{4\mu}{13}$$

$\therefore$
$$\theta = \tan^{-1}\left(\frac{4\mu}{13}\right)$$ **Ans.**

**33.** Writing equations of motion, we get




$$5g - T_2 = 5a_2 \qquad \text{...(i)}$$

$$\frac{(T_2 - T_1)R_2}{\frac{1}{2}M_2R_2^2} = \alpha_2 \qquad \text{...(ii)}$$

$$T_1 + f = M_1a_1 \qquad \text{...(iii)}$$

$$\frac{(T_1 - f)R_1}{\frac{1}{2}M_1R_1^2} = \alpha_1 \qquad \text{...(iv)}$$

$$a_1 = R_1\alpha_1 \qquad \text{...(v)}$$

$$a_1 + R_1\alpha_1 = R_2\alpha_2 \qquad \text{...(vi)}$$

$$R_2\alpha_2 = a_2 \qquad \text{...(vii)}$$

We have seven unknowns, $T_1$, $T_2$, $a_1$, $a_2$, $\alpha_1$, $\alpha_2$ and $f$ solving above equations, we get

$$a_2 = \frac{4}{11}g = 3.6 \text{ m/s}^2$$ **Ans.**

$$v = a_2 t = \frac{4gt}{11}$$ **Ans.**

**34.** Equations of motion are,

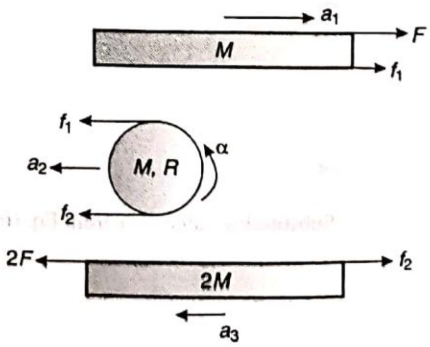
$$F + f_1 = Ma_1 \qquad \text{...(i)}$$

$$f_1 + f_2 = Ma_2 \qquad \text{...(ii)}$$

$$2F - f_2 = 2Ma_3 \qquad \text{...(iii)}$$

$$\alpha = \frac{(f_1 - f_2)R}{\frac{1}{2}MR^2}$$

or
$$\alpha = \frac{2(f_1 - f_2)}{MR} \qquad \text{...(iv)}$$



For no slipping condition,

$$a_2 + R\alpha = -a_1 \qquad \text{...(v)}$$

and

$$a_2 - R\alpha = a_3 \qquad \text{...(vi)}$$

We have six unknowns, $f_1$, $f_2$, $a_1$, $a_2$, $a_3$ and $\alpha$. Solving the above six equations, we get

$$a_1 = \frac{21F}{26M} \quad \text{and} \quad a_2 = \frac{F}{26M} \qquad \textbf{Ans.}$$

**35. Angular velocity :** From conservation of mechanical energy,

decrease in gravitational PE = increase in rotational KE

or

$$mgr \sin 60° + mg\,(2r \sin 60°) = \frac{1}{2}\left[\frac{3mr^2}{2} + m\,(2r)^2\right]\omega^2$$

$\therefore$

$$\frac{3\sqrt{3}mgr}{2} = \frac{11}{4}mr^2\omega^2$$

$\therefore$

$$\omega = \sqrt{\frac{6g\sqrt{3}}{11r}} \qquad \textbf{Ans.}$$

**Angular acceleration :** $\alpha = \dfrac{\tau}{I} = \dfrac{mgr\cos 60° + mg\,(2r\cos 60°)}{\left[\dfrac{3mr^2}{2} + m\,(2r)^2\right]} = \dfrac{\dfrac{3mgr}{2}}{\dfrac{11}{2}mr^2} = \dfrac{3g}{11r}$ **Ans.**

**36.**

$$\vec{\mathbf{a}}_B = \vec{\mathbf{a}}_0 + \vec{\mathbf{a}}_{B/0}$$

Here, $\vec{\mathbf{a}}_{B/0}$ has two components $a_t$ (tangential acceleration) and $a_n$ (normal acceleration)

$$a_t = r\alpha = (0.3)(5) = 1.5 \text{ m/s}^2$$

$$a_n = r\omega^2 = (0.3)(4)^2 = 4.8 \text{ m/s}^2$$

and

$$a_0 = 2 \text{ m/s}^2$$

$\therefore$

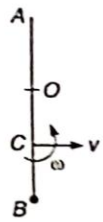
$$a_B = \sqrt{(\Sigma a_x)^2 + (\Sigma a_y)^2}$$

$$= \sqrt{(2 + 4.8\cos 45° - 1.5\cos 45°)^2 + (4.8\sin 45° + 1.5\sin 45°)^2}$$

$$= 6.21 \text{ m/s}^2$$

**37.** $C$ is the COM of $(M + m)$

$$BC = \left(\frac{M}{M+m}\right)\left(\frac{l}{2}\right) \quad \text{and} \quad OC = \left(\frac{m}{M+m}\right)\left(\frac{l}{2}\right)$$




From conservation of linear momentum,

$$(M + m)\,v = mv_0$$

or

$$v = \left(\frac{m}{M+m}\right)v_0 \qquad \text{...(i)}$$

From conservation of angular momentum about point $C$ we have,

$$mv_0(BC) = I\omega$$

or

$$\frac{mMv_0 l}{2\,(M+m)} = \left[m\left(\frac{M}{M+m}\right)^2\left(\frac{l}{4}\right)^2 + \frac{Ml^2}{12} + M\left(\frac{m}{M+m}\right)^2\left(\frac{l^2}{4}\right)\right]\omega$$

Putting $\frac{mv_0}{M+m} = v$ from Eq. (i), we have

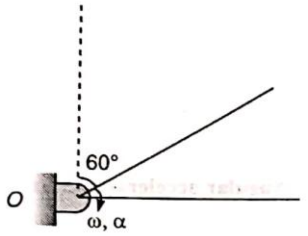
$$\frac{v}{\omega} = \frac{l}{6}\left[\frac{4m+M}{M+m}\right]$$

Now a point (say $P$) at a distance $x = \frac{v}{\omega}$, from $C$ (towards $O$) will be at rest. Hence, distance of point $P$ from boy at $B$ will be

$$BP = BC + x$$

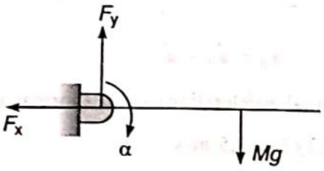
$$= \left(\frac{M}{M+m}\right)\left(\frac{l}{2}\right) + \frac{l}{6}\left[\frac{4m+M}{M+m}\right] = \frac{2l}{3}$$ **Ans.**

**38.** Let $\omega$ be the angular velocity and $\alpha$ the angular acceleration of rod in horizontal position. Then

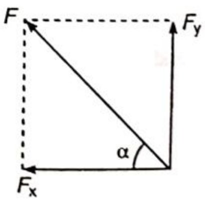



$$\alpha = \frac{(Mg)\frac{l}{2}}{\frac{Ml^2}{3}} = \frac{3}{2}\frac{g}{l} \quad \text{...(i)}$$

$$\frac{1}{2}\left(\frac{Ml^2}{3}\right)\omega^2 = Mg\frac{l}{4}$$




$\therefore$
$$\omega = \frac{3}{2}\cdot\frac{g}{l} \quad \text{...(ii)}$$




$$F_x = M\left(\frac{l}{2}\right)\omega^2 = M\left(\frac{l}{2}\right)\left(\frac{3g}{2l}\right) = \frac{3}{4}Mg$$

$$Mg - F_y = M(\alpha)\left(\frac{l}{2}\right)$$

$$F_y = Mg - \frac{3}{4}Mg = \frac{Mg}{4}$$

$\therefore$
$$F = \sqrt{F_x^2 + F_y^2} = \frac{\sqrt{10}}{4}Mg$$ **Ans.**

$$\tan\alpha = \frac{F_y}{F_x} = \frac{(Mg/4)}{(3Mg/4)} = \frac{1}{3}$$

$\therefore$
$$\alpha = \tan^{-1}\left(\frac{1}{3}\right)$$ **Ans.**

# *Solved Papers of NSEP*

# Solved Paper 2016

# NSEP

## National Standard Examination in Physics

*Conducted by: Homi Bhabha Centre for Science Education, India*

**Exam Held on** 27-11-2016

*NSEP is the first stage of selection of students in the Physics Olympiad Programme which is organised by the India Association of Physics Teachers (IAPT). Every student aspiring to go through successive stages of the programme must enroll for NSEP. NSEP is held at a large number of centres in the country.*

***Only one out of four options is correct***

**1.** The breakdown field for air is about $2 \times 10^6$ V/m. Therefore, the maximum charge that can be placed on a sphere of diameter 10 cm is

(a) $2.0 \times 10^{-4}$ C (b) $5.6 \times 10^{-7}$C
(c) $5.6 \times 10^{-2}$C (d) $2.0 \times 10^{2}$C

**2.** A wire in the shape of square frame carries a current $I$ and produces a magnetic field $B_s$ at its centre. Now, the wire is bent in the shape of a circle and carries the same current. If $B_c$ is the magnetic field produced at the centre of the circular coil, then $B_s / B_c$ is

(a) $8\pi^2$ (b) $\frac{8\pi^2}{\sqrt{2}}$ (c) $\frac{8\sqrt{2}}{\pi^2}$ (d) $8\pi\sqrt{2}$

**3.** A solid wooden block with a uniform cross-section is floating in water (density $\rho_w$) with a height $h_1$ below water. Now, a flat slab of stone is placed over the wooden block but the block still floats with a height $h_2$ below water. Afterwards the stone is removed from the top and pasted at the bottom of the wooden block. The wooden block now floats with a height $h_3$ under water. Therefore, the density of the stone is

(a) $\frac{h_2 - h_1}{h_3 - h_1} \rho_w$ (b) $\frac{h_2 - h_3}{h_2 - h_1} \rho_w$
(c) $\frac{h_2 - h_1}{h_2 - h_3} \rho_w$ (d) $\frac{h_3}{h_2 - h_1} \rho_w$

**4.** Two wires made of the same material, one thick and the other thin, are connected to form a composite wire. The composite wire is subjected to some tension. A wave travelling along the wire crosses the junction point.

The characteristic that undergoes a change at the junction point is

(a) Frequency only
(b) Speed of propagation only
(c) Wavelength only
(d) The speed of propagation as well as the wavelength

**5.** Ultraviolet light of wavelength 300 nm and intensity 1 W/m$^2$ falls on the surface of a photosensitive material.

If one per cent of the incident photons produce photoelectrons, then the number of photoelectrons emitted per second from an area of 1 cm$^2$ of the surface is nearly

(a) $1.51 \times 10^{13}$
(b) $1.51 \times 10^{12}$
(c) $4.12 \times 10^{13}$
(d) $2.13 \times 10^{11}$

**6.** At a certain height $h$ above the surface of the earth the change in the value of acceleration due to gravity ($g$) is the same as that at a depth $x$ below the surface. Assuming $h$ and $x$ to be enough small compared to the radius of the earth, $x : h$ is

(a) 1 : 1 (b) 2 : 1
(c) 1 : 2 (d) 1 : 4

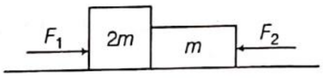
**7.** Two point masses $m_1$ and $m_2$ are connected at the ends of a light rigid rod of length $l$. The moment of inertia of the system about an axis through their centre of mass and perpendicular to the rod is

(a) $\frac{1}{2}\left(\frac{m_1 m_2}{m_1 + m_2}\right) l^2$ (b) $\left(\frac{m_1 m_2}{m_1 + m_2}\right) l^2$
(c) $(m_1 + m_2)\, l^2$ (d) $[m_1^2 + m_2^2]\left(\frac{m_1 + m_2}{m_1 m_2}\right) l^2$

**8.** Two particles of masses $m$ and $M$ are initially at rest and infinitely separated. At a later instant when they are at a finite distance $d$ from each other, their relative velocity of approach is

(a) $\left[\frac{Gm}{2d}\right]^{\frac{1}{2}}$ (b) $\left[\frac{2G(m + M)}{d}\right]^{\frac{1}{2}}$
(c) $\left[\frac{G(m + M)}{2d}\right]^{\frac{1}{2}}$ (d) $\left[\frac{2(GM)}{d}\right]^{\frac{1}{2}}$

**9.** Two blocks of masses $m$ and $2m$ are placed on a smooth horizontal surface as shown. In the first case only a force $F_1$ is applied from left. Later on only a force $F_2$ is applied from right. If the force acting at the interface of the two blocks in the two cases is the same, then $F_1 : F_2$ is




(a) 1 : 1 (b) 1 : 2 (c) 2 : 1 (d) 1 : 3

**10.** A ball $A$ of mass 1 kg moving at a speed of 5 m/s strikes tangentially another ball $B$ initially at rest. The ball $A$ then moves at right angles to its initial direction at a speed of 4 m/s.

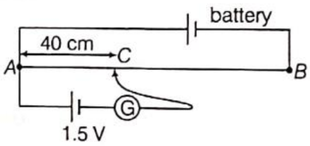
If the collision is elastic, the mass (in kg) of ball $B$ and its momentum after collision (in kg-m/s) respectively are approximately

(a) 1.2 and 1.8 (b) 2.2 and 3.3
(c) 4.6 and 6.4 (d) 6.2 and 9.1

**Directions (Q. Nos. 11-14)** *These questions are based on the following paragraph.*

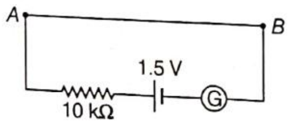
*A nichrome with AB, 100 cm long and of uniform cross-section is mounted on a meter scale, the points A and B coinciding with 0 cm and 100 cm marks respectively. The wire has a resistance $S = 50\,\Omega$. Any point C along this wire, between A and B is called a variable point to which one end of an electrical element is connected. In the following questions, this arrangement will be referred to as wire AB.*

**11.** The emf of a battery is determined using the following circuit with wire $AB$. The galvanometer shows zero deflection when one of its terminals is connected to point $C$. If the internal resistance of the battery is $4\,\Omega$, its emf is




(a) 3.75 V (b) 4.05 V
(c) 2.50 V (d) 9.0 V

**12.** In the adjacent arrangement it is found that deflection in the galvanometer is 10 divisions. Also, the voltage across the wire $AB$ is equal to that across the galvanometer. Therefore, the current sensitivity of the galvanometer is $d$ about

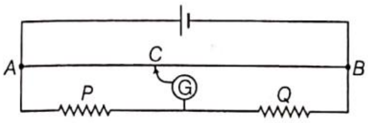



(a) 0.050 div/μA (b) 0.066 div/μA
(c) 0.010 div/μA (d) Data insufficient

**13.** The wire $AB$ is now a part of the adjacent circuit. With the resistors $P = 50\,\Omega$ and $Q = 100\,\Omega$, the null point is obtained at $C$ where $AC = 33$ cm. When the resistors are interchanged, the null point is found at $C$ with $AC = 67$ cm.

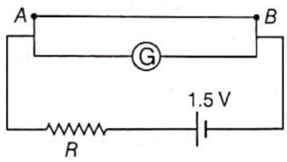
The systematic error in this experiment seems to be due to non-coincidence of $A$ and $B$ with cm mark and 100 cm mark respectively.

If these end errors and equivalent to '$a$' cm and '$b$' cm respectively, then they are




(a) 0 and 1 (b) 1 and 0
(c) 0.33 and 0.33 (d) 1 and 1

**14.** In the adjacent circuit, a resistance $R$ is used. Initially with wire $AB$ not in the circuit, the galvanometer shows a deflection of $d$ divisions. Now, the wire $AB$ is connected parallel to the galvanometer and the galvanometer shows a deflection nearly $d/2$ divisions. Therefore,




(a) $R = G$ (b) $R << G$
(c) $R >> G$ (d) $R = \frac{SG}{S+G}$

**15.** Consider a relation connecting three physical quantities $A, B$ and $C$ given by $A = B^n C^m$. The dimensions of $A$, $B$ and $C$ are [LT], [$L^2T^1$] and [$LT^2$] respectively.
Therefore, the exponents $n$ and $m$ have values
(a) 2/3 and 1/3 (b) 2 and 3
(c) 4/5 and –1/5 (d) 1/5 and 3/5

**16.** Two identical rooms in a house are connected by an open doorway. The temperatures in the two rooms are maintained at two different values.
Therefore,
(a) the room with higher temperature contains more amount of air
(b) the room with lower temperature contains more amount of air
(c) Both the rooms contain the same amount of air
(d) the room with higher pressure contains more amount of air

**17.** A vibrator of frequency $f$ is placed near one end of a long cylindrical tube. The tube is fitted with a movable piston at the other end. An observer listens to the sound through a side opening. As the piston is moved through 8.75 cm, the intensity of sound recorded by the observer changes from a maximum to a minimum.
If the speed of sound in air is 350 m/s, then the frequency $f$ is
(a) 500 Hz (b) 1000 Hz
(c) 2000 Hz (d) 4000 Hz

**18.** A heavy metal block is dragged along a rough horizontal surface at a constant speed of 20 km/h. The coefficient of friction between the block and the surface is 0.6. The block is made of a material whose specific heat 0.1 cal/g-°C and absorbs 25% of heat generated due to friction.
If the block is dragged from 10 min, the rise in temperature of the block is about ($g = 10\ \text{m/s}^2$)
(a) 12°C
(b) 50°C
(c) 210°C
(d) Data insufficient

**19.** A gas is made to undergo a change of state from an initial state to a final state along different paths by adiabatic process only. Therefore,
(a) the work done is different for different paths.
(b) the work done is the same for all paths.
(c) there is no work as there is no transfer of energy.
(d) the total internal energy of the system will not change.

**20.** Vectors **A**, **B**, **C** lie in $XY$-plane and their resultant is **R**. The magnitudes of **A**, **B** and **R** are 100, 200 and 200 respectively. The angles made by these vectors with the positive direction of $X$-axis are 60°, 150° and 90° respectively.
Therefore, the magnitude and the angle made by **C** with positive direction of $X$-axis respectively are
(a) 75, 315° (b) 110, 45°
(c) 156, 240° (d) 124, 6.2°

**21.** Two particles $A$ and $B$ are situated 10 m apart along $X$-axis, $B$ being farther right of $A$, at $t = 0$. Particle $A$ is moving at 0.75 m/s parallel to $+Y$-axis while $B$ at 1 m/s along $-X$-axis. After a time $t$, they come closest to each other. Therefore, $t$ is

(a) 4.8 s (b) 6.4 s
(c) 6.0 s (d) 3.2 s

**22.** Out of the following differential equations, one that correctly represents the motion of a second's pendulum is

(a) $\frac{d^2x}{dt^2} + \frac{x}{\pi} = 0$ (b) $\frac{d^2x}{dt^2} + \frac{x}{\pi^2} = 0$
(c) $\frac{d^2x}{dt^2} + \pi x = 0$ (d) $\frac{d^2x}{dt^2} + \pi^2 x = 0$

**23.** A block of mass 2 kg drops vertically from a height of 0.4 m onto a spring whose force constant $K$ is 1960 N/m.

Therefore, the maximum compression of the spring is

(a) 0.40 m (b) 0.25 m
(c) 0.80 m (d) 0.1 m

**24.** Two blocks of masses $m_1 = 8$ kg and $m_2 = 7$ kg are connected by a light string passing over a light frictionless pulley. The mass $m_1$ is at rest on the inclined plane and mass $m_2$ hangs vertically. The angle of inclination is 30°.

Therefore, the force of friction acting on $m_1$ is

(a) 30 N up the plane (b) 30 N down the plane
(c) 40 N up the plane (d) 40 N down the plane

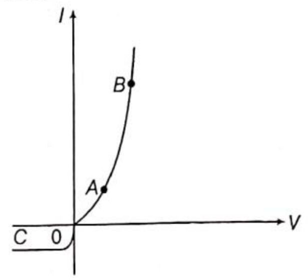
**25.** Two factories are sounding their sirens at 400 Hz each. A man walks from one factory towards the other at a speed of 2 m/s. The speed of sound is 320 m/s.

The number of beats heard per second by the man is

(a) 6 (b) 5
(c) 2.5 (d) 7.5

**26.** The adjacent figure shows $I$-$V$ characteristics of a silicon diode. In this connection three statements are made-(I) the region $OC$ corresponds to reverse bias of the diode, (II) the voltage at point $A$ is about 0.2 V and (III) different scales have been used along +ve and –ve directions of $Y$-axis.

Therefore,



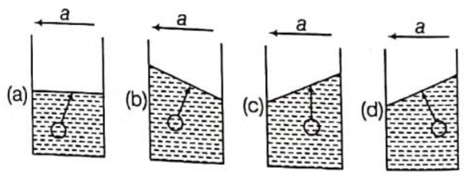

(a) only statement (I) is correct
(b) only statements (I) and (II) are correct
(c) only statements (I) and (III) are correct
(d) all statements (I), (II) and (III) are correct

**27.** Two identical lenses made of the same material of refractive index 1.5 have the focal length 12 cm. These lenses are kept in contact and immersed in a liquid of refractive index 1.35. The combination behave as

(a) convex lens of focal length 27 cm
(b) concave lens of focal length 6 cm
(c) convex lens of focal length 9 cm
(d) convex lens of focal length 6 cm

**28.** A cup of water is placed in a car moving at a constant acceleration $a$ to the left. Inside the water is a small air bubble.

The figure that correctly shows the shape of the water surface and the direction of motion of the air bubble is




**29.** A sphere of radius $R$ made up of styrofoam (light polystyrene material) has a cavity of radius $R/2$. The centre of the cavity is situated at a distance of $R/2$ from the centre of the styrofoam sphere. The cavity is filled with a solid material of density five times that of styrofoam.

Now, the centre of mass is seen to be located at a distance $x$ from the centre of styrofoam sphere, therefore $x$ is

(a) $R/2$ (b) $R/3$ (c) $R/4$ (d) $R/6$

**30.** A resistor $R$ is connected to a parallel combination of two identical batteries each with emf $E$ and an internal resistance $r$.
The potential drop across the resistance $R$ is

(a) $\frac{2ER}{2R+r}$ (b) $\frac{ER}{R+2r}$

(c) $\frac{ER}{2R+r}$ (d) $\frac{2ER}{R+2r}$

**31.** The critical angle between a certain transparent medium and air is $\phi$. A ray of light travelling through air enters the medium at an angle of incidence equal to its polarising angle $\theta$. Therefore, the angle of refraction is

(a) $\tan^{-1}(\sin\theta)$ (b) $\tan^{-1}(\sin\phi)$

(c) $\sin^{-1}(\tan\theta)$ (d) $\sin^{-1}(\tan\phi)$

**32.** If a copper wire is stretched to make its radius decrease by 0.1%, the percentage changes in its resistance is approximately

(a) – 0.4 % (b) + 0.8%

(c) + 0.4 % (d) + 02 %

**33.** Consider a manual camera with a lens having a focal length of 5 cm.

It is focused at infinity. For catching the picture of an object at a distance of 30 cm, one would

(a) move the lens out by about 1 cm

(b) move the lens out by about 5 cm

(c) move the lens in by about 1 cm

(d) find it impossible to catch the picture

**34.** Initially interference is observed with the entire experimental set up inside a chamber filled with air. Now, the chamber is evacuated. With the same source of light used, a careful observer will find that

(a) the interference pattern is almost absent as it is very much diffused

(b) there is no change in the interference pattern

(c) the fringe width is slightly decreased

(d) the fringe width is slightly increased.

**35.** Two identical loudspeakers, placed closed to each other inside a room, are supplied with the same sinusoidal voltage. One can imagine a pattern around the loudspeakers with areas of increased and decreased sound intensity alternately located.

Which of the following actions will NOT change the locations of these areas?

(a) Moving one of the speakers

(b) Changing the amplitude of the signal voltage

(c) Changing the frequency of the signal voltage

(d) Replacing the air in the room with a different gas

**36.** A particle at rest explodes into two fragments of masses $m_1$ and $m_2$ $(m_1 > m_2)$ which move apart with non-zero velocities.
If $\lambda_1$ and $\lambda_2$ are their de-Broglie wavelengths respectively, then

(a) $\lambda_1 > \lambda_2$ (b) $\lambda_1 < \lambda_2$

(c) $\lambda_1 = \lambda_2$ (d) Data insufficient

**37.** Two particles of masses $m_1$ and $m_2$ carry identical charges. Starting from rest they are accelerated through the same potential difference. Then they enter into a region of uniform magnetic field and move along circular paths $R_1$ and $R_2$ respectively.
Therefore, the ratio of their masses $m_1 : m_2$ is

(a) $R_1 : R_2$ (b) $R_1^2 : R_2^2$ (c) $R_2^2 : R_1^2$ (d) $\sqrt{R_1} : \sqrt{R_2}$

**38.** A fixed horizontal wire $M$ carries 200 A current. Another wire $N$ running parallel to $M$ carries a current $I$ and remains suspended in a vertical plane below $M$ at a distance of 20 mm. Both the wires have a linear mass density $10^{-2}$kg/m. Therefore, the current $I$ is

(a) 20 A (b) 4.9 A (c) 49 A (d) 200 A

**39.** An upolarised light of intensity 32 W/m$^2$ passes through three polarisers, such that the transmission axis of last polariser is crossed with that of the first.

If the intensity of emergent light is 3 W/m$^2$, then the angle between the transmission axes of the first two polarisers is

(a) 30° (b) 19° (c) 45° (d) 90°

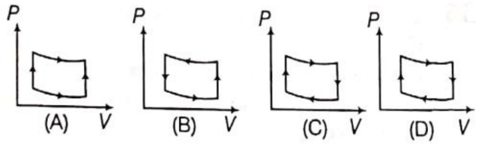
**40.** An electron in injected directly towards the centre of a large metal plate having a uniform surface charge density of $-2.0 \times 10^{-6}$C/m$^2$. The initial kinetic energy of the electron is $1.6 \times 10^{-17}$ J. The electron is observed to stop as it just reaches the plate. Therefore, the distance between the plate and the point from where the electron was injected is

(a) $4.4 \times 10^{-4}$ m (b) 4.4 m

(c) $4.4 \times 10^{-2}$ m (d) Data insufficient

**41.** Graphs (drawn with the same scale) showing the variation of pressure with volume for a certain gas undergoing four different cyclic processes A,B,C and D are given below.

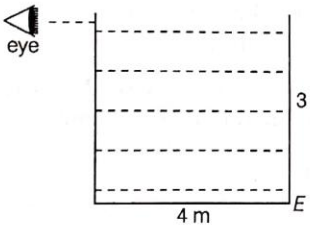
The cyclic process in which the gas performs the greatest amount of work is




(a) A
(b) B
(c) C
(d) D

**42.** A rectangular metal tank filled with a certain liquid is as shown in the figure. The observer, whose eye is in level with the top of the tank, can just see the corner $E$ of the tank.

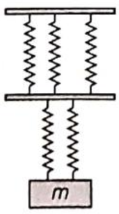
Therefore, the refractive index of the liquid is




(a) 1.67
(b) 1.50
(c) 1.33
(d) 1.25

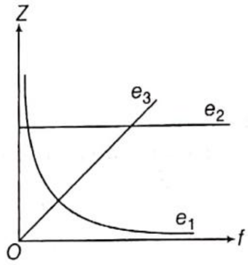
**43.** As shown in the figure, a block of mass $m$ is suspended from a support with the help of a system of identical springs. The force constant of each spring is $k$.

Therefore, the frequency of oscillations of the block is




(a) $\frac{1}{2\pi}\sqrt{\frac{3k}{2m}}$
(b) $\frac{1}{2\pi}\sqrt{\frac{2k}{3m}}$
(c) $\frac{1}{2\pi}\sqrt{\frac{5k}{6m}}$
(d) $\frac{1}{2\pi}\sqrt{\frac{6k}{5m}}$

**44.** The impedance ($Z$) of three electrical components $e_1$, $e_2$ and $e_3$ has frequency ($f$) dependence as shown by the following three curves. Three components $e_1, e_2, e_3$ are respectively.




(a) $R, L, C$
(b) $R, C, L$
(c) $L, R, C$
(d) $C, R, L$

**45.** The half-life period of a radioactive element $E_1$ is equal to the mean lifetime of another radioactive element $E_2$. Initially both the elements have the same number of atoms. Therefore,

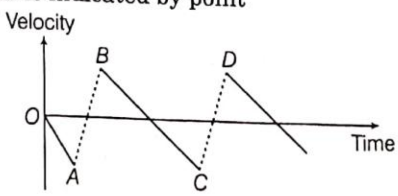
(a) $E_2$ will decay faster
(b) $E_1$ will decay faster
(c) $E_1$ and $E_2$ will decay at the same rate
(d) Data insufficient

**46.** A simple pendulum has a bob of mass $m$ and a light string of length $l$. The string is replaced by a uniform rod of mass $m$ and of the same length $l$. The time period of this pendulum is

(a) $2\pi(l/g)^{1/2}$
(b) $2\pi(8l/9g)^{1/2}$
(c) $2\pi(9l/8g)^{1/2}$
(d) $2\pi(2l/3g)^{1/2}$

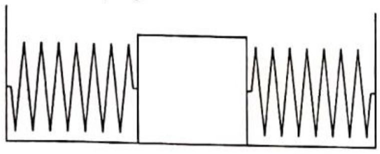
**47.** A tennis ball is released from a height and allowed to fall onto a hard surface. The adjacent graph shows the variation of velocity of the ball with time from the instant of release.

The point of upward maximum velocity of the ball is indicated by point



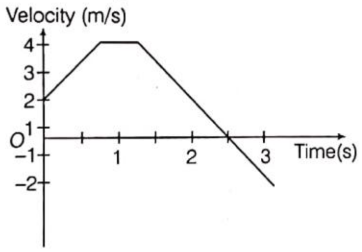

(a) $A$
(b) $B$
(c) $C$
(d) $D$

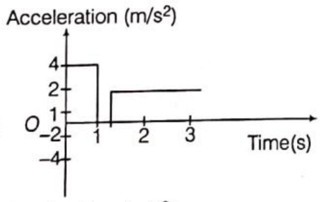
**48.** The diagram shows an oscillating block connected to two identical springs. The frequency of oscillations can be increased substantially by

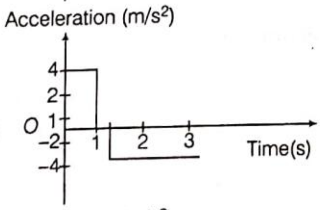


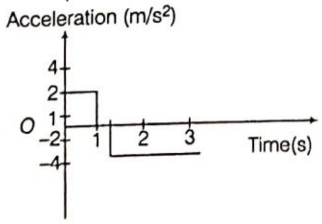
(a) increasing the amplitude of the oscillations
(b) fixing an extra mass to the block
(c) using softer pair of springs
(d) using harder pair of springs

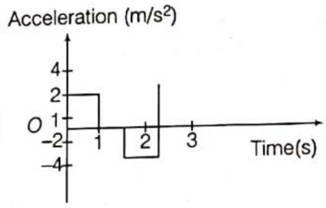
**49.** The variation of velocity with time of a toy car moving along a straight line is as in adjacent figure. Which of the following graph correctly represents the variation of acceleration with time for the toy car?




(a)



(b)



(c)



(d)



**50.** An AC source (sinusoidal source with frequency 50 Hz) is connected in series with a rectifying diode, a 100 Ω resistor, a 1000 μF capacitor and a milliammeter. After some time the milliammeter reads zero. The voltage measured across the capacitor with a DC voltmeter is

(a) the peak voltage of the AC source
(b) rms voltage of the AC source
(c) average voltage of the AC source over a half-cycle
(d) average voltage of the AC source over a full cycle

**51.** The frequency of the sound produced by a siren increases from 400 Hz to 1200 Hz while its amplitude remains the same. Therefore, the ratio of the intensity of the 1200 Hz wave to that of the 400 Hz wave is

(a) 1 : 1 (b) 3 : 1 (c) 1 : 9 (d) 9 : 1

**52.** The fundamental frequency of the output of a bridge rectifier driven by AC mains is

(a) 50 Hz (b) zero
(c) 100 Hz (d) 25 Hz

**53.** The force of attraction between the positively charged nucleus and the electron in a hydrogen atom is given by $f = k\frac{e^2}{r^2}$. Assume that the nucleus is fixed. The electron, initially moving in an orbit of radius $R_1$ jumps into an orbit of smaller radius $R_2$. The decrease in the total energy of the atom is

(a) $\frac{ke^2}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$ (b) $\frac{ke^2}{2}\left(\frac{R_1}{R_2^2}-\frac{R_2}{R_1^2}\right)$

(c) $\frac{ke^2}{2}\left(\frac{1}{R_2}-\frac{1}{R_1}\right)$ (d) $\frac{ke^2}{2}\left(\frac{R_2}{R_1^2}-\frac{R_1}{R_2^2}\right)$

**54.** It is observed that some of the spectral lines in hydrogen spectrum have wavelengths almost equal to those of the spectral lines in $He^+$ ion. Out of the following the transitions in $He^+$ that will make this possible is

(a) $n = 3$ to $n = 1$ (b) $n = 6$ to $n = 4$
(c) $n = 5$ to $n = 3$ (d) $n = 3$ to $n = 2$

**Directions (Q. Nos. 55-60)** *These questions are based on the following paragraph.*

*A wheel of a car is made up of two parts (1) the central metal rim and (2) the rubber tyre. The width of the tyre W =16.5 cm and height h =10.7 cm. The rim overlaps the tyre. The total weight of the car is 1500 kg distributed evenly. The tyres are inflated with air to a pressure 2.0 kg/cm². The density of air at pressure of 1.0 kg/cm² and at room temperature equals 1.29 g/litre. The outer diameter of the tyre is 55.4 cm and that of the rim is 40 cm.*

*Ignore the thickness of rubber and use the dimensions given here. Note that the units mentioned above are conventional units used in everyday life.*

**55.** Consider the following two statements about a tyre of a car.

**Statement A** The horizontal road surface is exactly tangential to the tyre.

**Statement B** The tyre is inflated with excess pressure.

(a) Statement A is the result of Statement B.
(b) Statement B cannot be true.
(c) Statement A cannot be true.
(d) Neither of the Statement A and B is true.

**56.** The left side front tyre was observed to be in contact with the road over a length $L$ cm. The value of $L$ is

(a) 8.85 cm
(b) 9.35 cm
(c) 11.36 cm
(d) 10.35 cm

**57.** When five persons occupy the seats $L$ increases by 2.5 cm. The average weight of a person is

(a) 66 kg (b) 60 kg
(c) 62 kg (d) 64 kg

**58.** If five persons occupy the seats, the centre of the wheel is lowered by about

(a) 1 mm (b) 2 mm
(c) 3 mm (d) 4 mm

**59.** The mass of air in a tyre is about

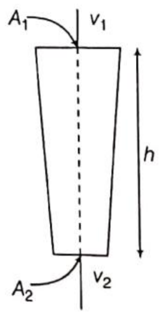
(a) 24 g
(b) 49 g
(c) 32 g
(d) 62 g

**60.** The tyres of racing cars are very wide. Their width is nearly three times the above value. This large width is for

(a) stability and acceleration
(b) stream lining and acceleration
(c) stream lining and stability
(d) stream lining, stability and acceleration

***One or more than one options are correct***

**61.** Water is flowing through a vertical tube with varying cross-section as shown. The rate of flow is 52.5 mL/s. Given that speed of flow $v_1 = 0.35$ m/s and area of cross-section $A_2 = 0.5$ cm². Which of the following is/are true?




(a) $A_1 = 1.0\text{cm}^2, v_2 = 0.70$ m/s
(b) $A_1 = 1.5\text{cm}^2, v_2 = 1.05$ m/s
(c) $h = 5$ cm
(d) $h = 10$ cm

**62.** A simple laboratory power supply consists of a transformer, bridge rectifier and a filter capacitor. It drives a suitable load. If due to some reason one of the diodes in the rectifier circuit becomes open, then

(a) output voltage of power supply falls to zero
(b) output voltage of power supply decreases to some non-zero value
(c) AC ripple in the output increases
(d) ripple frequency decreases

**63.** Circuit $A$ is a series $L$-$C$-$R$ circuit with $C_A = C$ and $L_A = L$. Another circuit $B$ has $C_B = 2C$ and $L_B = L/2$. Both the circuits have the same resistance and the capacitor and the inductance are assumed to be ideal components. Each of the circuits is connected to the same sinusoidal voltage source.

Therefore,

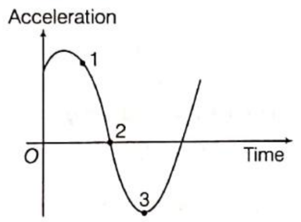
(a) both the circuits have the same resonant frequency

(b) both the circuits carry the same peak current

(c) resonance curve for circuit $A$ is more sharp than that for circuit $B$

(d) resonance curve for circuit $B$ is more sharp than that for circuit $A$

**64.** The variation of acceleration with time for a particle performing simple harmonic motion along straight line is as in adjacent figure. Therefore,




(a) the particle has a non-zero displacement initially

(b) the displacement of the particle at point 1 is negative

(c) the velocity of the particle at point 2 is positive

(d) the potential energy at point 3 is maximum

**65.** Which of the following physical quantities have dimensions identical to each other?

(a) The Young's modulus $Y$

(b) $\varepsilon_0 E^2$ where, $E$ is the electric field intensity and $\varepsilon_0$ is the permittivity of free space

(c) $\frac{B^2}{\mu_0}$ where, $B$ is the magnetic field and $\mu_0$ is the permeability of free space

(d) $kT$ where, $k$ is Boltzmann's constant and $T$ is the absolute temperature

**66.** A small ball bearing is released at the top of a long vertical column of glycerin of height $2h$. The ball bearing falls through a height $h$ in a time $t_1$ and then the remaining height with the terminal velocity in time $t_2$. Let $W_1$ and $W_2$ be the work done against viscous drag over these heights. Therefore,

(a) $t_1 < t_2$ (b) $t_1 > t_2$

(c) $W_1 = W_2$ (d) $W_1 < W_2$

**67.** A particle moves in $XY$-plane according to the relations $x = kt$ and $y = kt\,(1 - pt)$ where, $k$ and $p$ are positive constants and $t$ is time. Therefore,

(a) the trajectory of the particle is a parabola

(b) the particle has a constant velocity along $X$-axis

(c) the force acting on the particle remains in the same direction even if $k$ and $p$ are negative constants

(d) the particle has a constant acceleration along $-Y$-axis

**68.** A charge $q$ is situated at the origin. Let $E_A$, $E_B$ and $E_C$ be the electric field at the points $A\,(2, -3, -1)$, $B\,(-1, -2, 4)$ and $C\,(2, -4, 1)$. Therefore,

(a) $E_A \perp E_B$

(b) no work is done in moving a test charge $q_0$ from $B$ to $C$

(c) $2|E_A| = 3|E_B|$

(d) $E_B = -E_C$

**69.** A uniform spherical charge distribution of radius $R$ produces electric field $E_1$ and $E_2$ at two points at distances $r_1$ and $r_2$ respectively from the centre of the distribution.

Out of the following, the possible expression/s for $\frac{E_1}{E_2}$ is /are

(a) $\frac{r_1}{r_2}$ (b) $\left[\frac{r_1}{r_2}\right]^2$ (c) $\frac{R^3}{r_1^2 r_2}$ (d) $\frac{r_1 r_2^3}{R^3}$

**70.** A metallic wire of length $l$ is held between two supports under some tension. The wire is cooled through $\theta°$. Let $Y$ be the Young's modulus, $\rho$ the density and $\alpha$ the thermal coefficient of linear expansion of the material of the wire. Therefore, the frequency of oscillations of the wire varies as

(a) $\sqrt{Y}$ (b) $\sqrt{\theta}$

(c) $\frac{1}{l}$ (d) $\sqrt{\frac{\alpha}{\rho}}$

# Detailed Solutions

**1.** (b) Given, diameter of sphere $(d) = 10\,\text{cm}$

$$= 10 \times 10^{-2}\text{m}$$

Radius of sphere $(r) = \frac{10 \times 10^{-2}}{2} = 5 \times 10^{-2}\text{m}$

Breakdown field for air $(E) = 2 \times 10^6$ V/m

We know that, $E = \frac{Kq}{r^2}$

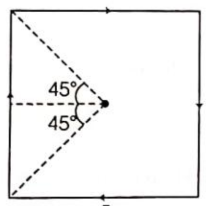
$$\Rightarrow \quad q = \frac{E \times r^2}{K}$$

$$\Rightarrow \quad q = \frac{2 \times 10^6 \times (5 \times 10^{-2})^2}{9 \times 10^9}$$

$$\Rightarrow \quad q = \frac{2 \times 10^6 \times 25 \times 10^{-4}}{9 \times 10^9}$$

$$\Rightarrow \quad q = \frac{50}{9} \times 10^{-7} = 5.6 \times 10^{-7}\,\text{C}$$

**2.** (c) According to question, we can draw the following diagram.




Magnetic field at the centre of square is given by

$$B_s = 4\left[\frac{\mu_0 I}{4\pi \frac{L}{2}}\sqrt{2}\right] \quad \ldots\text{(i)}$$

Here, $I$ = current flowing through the square

$L$ = side of square

$$B_s = \frac{2\mu_0 I\sqrt{2}}{\pi L}$$

Magnetic field at the centre of circular coil is given by

$$B_c = \frac{\mu_0 I}{2R} \quad \ldots\text{(ii)}$$

The circumference of circular coil is equal to the circumference of square.

$$\therefore \quad 2\pi R = 4L$$

$$R = \frac{4L}{2\pi} \quad \ldots\text{(iii)}$$

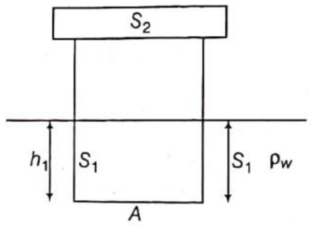
Substituting the value of Eq. (iii) in Eq. (ii), we get

$$B_c = \frac{\mu_0 I}{2\left(\frac{4L}{2\pi}\right)} = \frac{\mu_0 I\,\pi}{4L} \quad \ldots\text{(iv)}$$

After dividing Eq. (i) by Eq. (ii), we get

$$\frac{B_s}{B_c} = \frac{2\sqrt{2}}{\pi^2} \cdot 4 = \frac{8\sqrt{2}}{\pi^2}$$

**3.** (c) According to question, we can draw the following diagram.




From figure, $\rho_1 HAg = \rho_w h_1 Ag$

$$h_1 = \frac{\rho_1 H}{\rho_w} \quad \ldots\text{(i)}$$

Similarly, $\rho_1 HAg + \rho_2 Vg = \rho_w Ah_2 g$

$$h_2 = \frac{\rho_1 AH + \rho_2 V}{\rho_w A} \quad \ldots\text{(ii)}$$

Now, $\rho_w Vg + \rho_w Ah_3 g = \rho_w\ Ah_2 g$

$$V = A(h_2 - h_3) \quad \ldots\text{(iii)}$$

From Eq. (ii), $h_2 = \frac{\rho_1 AH + \rho_2 A(h_2 - h_3)}{\rho_w A}$

$$h_2 = \frac{(h_1\rho_w) + \rho_2(h_2 - h_3)}{\rho_w}$$

$$h_2\rho_w - h_1\rho_w = \rho_2(h_2 - h_3)$$

$$\rho_2 = \frac{(h_2 - h_1)}{h_2 - h_3}\rho_w$$

**4.** (d) We know that, $v \propto \sqrt{\frac{T}{\mu}}$

$$\Rightarrow \quad \mu_2 > \mu_1$$

Thus, speed of propagation as well as wavelength changes.

**5.** (b) Given, wavelength of ultraviolet light,

$$(\lambda) = 300\,\text{nm} = 300 \times 10^{-9}\,\text{m}$$

Intensity of ultraviolet light $(I) = 1\,\text{W/m}^2$

Area $(A) = 1\,\text{cm}^2 = 1 \times 10^{-4}\,\text{m}^2$

According to question,
Number of photons,

$$n = \frac{IA}{(hc/\lambda)}\left(\frac{\lambda}{100}\right)$$

[Here, $h$ = Planck constant]

$$= \frac{1 \times 1 \times 10^{-4} \times 300 \times 10^{-9}}{(6.626 \times 10^{-34})(3 \times 10^{8})} \times \frac{1}{100}$$

$$= (0.151) \times 10^{13}$$

$$= 1.51 \times 10^{12}$$

**6.** (b) The gravitational acceleration at height $h$ from the surface of earth is given by

$$g'_h = g\left[1 - \frac{2h}{R}\right] \quad \ldots\text{(i)}$$

The gravitational acceleration at depth $x$ from the surface of earth is given by

$$g'_d = g\left[1 - \frac{x}{R}\right] \quad \ldots\text{(ii)}$$

According to question,

$$g'_h = g'_d$$

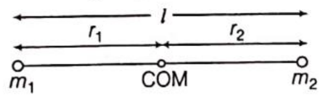
$$g\left[1 - \frac{2h}{R}\right] = g\left[1 - \frac{x}{R}\right]$$

$$\Rightarrow \quad \frac{2h}{R} = \frac{x}{R}$$

$$\Rightarrow \quad 2h = x$$

$$\Rightarrow \quad \frac{x}{h} = \frac{2}{1} \text{ or } 2:1$$

**7.** (b) According to question,




$$m_1 r_1 = m_2 r_2$$

Now, we can write that

$$r_1 = \frac{m_2 l}{m_1 + m_2}$$

$$r_2 = \frac{m_1 l}{m_1 + m_2}$$

Now, the moment of inertia of the system about an axis through their centre of mass and perpendicular to the rod is

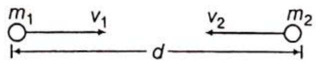
$$I = m_1 r_1^2 + m_2 r_2^2$$

$$I = m_1 r_1 (r_1 + r_2)$$

$$I = m_1\left(\frac{m_2 l}{m_1 + m_2}\right) l \quad [\because r_1 + r_2 = l]$$

$$I = \frac{m_1 m_2 l^2}{m_1 + m_2}$$

**8.** (b)




By conservation of linear momentum

$$m_1 v_1 = m_2 v_2$$

$$v_2 = \frac{m_1 v_1}{m_2}$$

$$v_2 = \frac{m v_1}{M} \quad \ldots\text{(i)}$$

Now, $\frac{1}{2} m v_1^2 + \frac{1}{2} m v_2^2 = \frac{GMm}{d}$

From Eq. (i)

$$\frac{GMm}{d} = \frac{1}{2} m v_1^2 + \frac{1}{2} M\left(\frac{m v_1}{M}\right)^2$$

$$\frac{GMm}{d} = \frac{1}{2} m v_1^2 \left(1 + \frac{m}{M}\right)$$

$$v_1^2 = \frac{2GM^2}{d(m+M)}$$

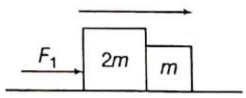
$$v_1 = \sqrt{\frac{2GM^2}{d(m+M)}} = M\sqrt{\frac{2G}{d(m+M)}}$$

Similarly, $\quad v_2 = m\sqrt{\frac{2G}{d(m+M)}}$

Relative velocity $= v_1 + v_2 = \left[\frac{2G(m+M)}{d}\right]^{1/2}$

**9.** (c) According to question,

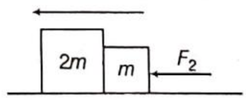
**Case I** When force $F_1$ is applied from left.




Now, acceleration, $a = \frac{F_1}{2m + m} = \frac{F_1}{3m}$

$$F'_1 = ma = \frac{F_1}{3m}(m) \quad \ldots\text{(i)}$$

**Case II** When force $F_2$ is applied from right.




Acceleration, $a = \frac{F_2}{3m}$

$$F'_2 = 2ma = \frac{F_2}{3m}(2m) \quad \ldots\text{(ii)}$$

Now, $\quad F'_1 = F'_2$

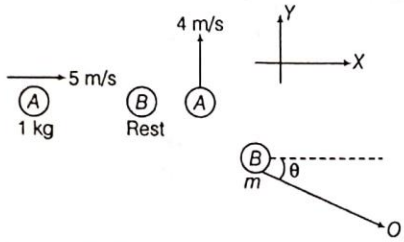
$$\frac{F_1(m)}{3m} = \frac{F_2(2m)}{3m}$$

$$\frac{F_1}{F_2} = \frac{2}{1} \text{ or } 2:1$$

**10.** (c) Given, mass of ball $A = 1\,\text{kg}$

Speed of ball $A = 5\,\text{m/s}$

According to question,




Conservation of momentum along $Y$-axis,

$$m_A v_A = M v_2$$
$$1 \times 4 = M v_2$$
$$4 = mv \sin\theta \quad \ldots\text{(i)}$$

Conservation of momentum along $X$-axis

$$1 \times 5 + m \times 0 = 1 \times 0 + m(v\cos\theta)$$
$$5 = mv\cos\theta \quad \ldots\text{(ii)}$$

From Eqs. (i) and (ii),

$$5^2 + 4^2 = m^2v^2\sin^2\theta + m^2v^2\cos^2\theta$$
$$\Rightarrow \quad 5^2 + 4^2 = m^2v^2(\sin^2\theta + \cos^2\theta)$$
$$\Rightarrow \quad mv = \sqrt{5^2 + 4^2}$$
$$\Rightarrow \quad = \sqrt{25 + 16}$$
$$\Rightarrow \quad mv = \sqrt{41} \approx 6.4\,\text{kg m/s}$$

Now, from conservation of kinetic energy

$$\frac{1}{2} m_A v_A^2 = \frac{1}{2} m'_A v'_A + \frac{1}{2}\frac{p^2}{m}$$
$$\Rightarrow \quad \frac{1}{2}(1)(5)^2 = \frac{1}{2}(1)(4)^2 + \frac{1}{2}\frac{p^2}{m}$$
$$\Rightarrow \quad \frac{25}{2} = 8 + \frac{41}{2m} \quad [\because P = mv]$$
$$\Rightarrow \quad \frac{41}{2m} = 4.5$$
$$\Rightarrow \quad m = \frac{41}{9} = 4.6\,\text{kg}$$

**11.** (b) Given, potential across 40 cm = 1.5 V

$$R = 50\,\Omega$$

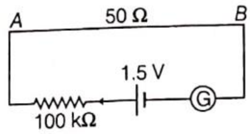
Now, potential drop across the wire $AB$.

$$V_{AB} = \frac{1.5}{40} \times 100 = \frac{15}{4}\,\text{V}$$

Terminal voltage across the battery is

$$V_{AB} = \frac{E(R)}{R + r} \Rightarrow \frac{15}{4} = \frac{E(50)}{50 + 4}$$
$$E = \frac{15 \times 54}{50 \times 4} = 4.05\,\text{V}$$

**12.** (b)




Given, number of divisions on galvanometer

$$(n) = 10\,\text{div}$$

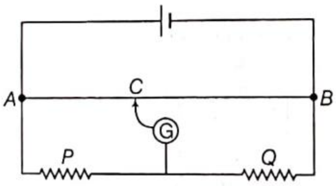
We know that,

$$\text{Current } (i) \approx \frac{E}{R} = \frac{1.5}{10 \times 10^3}$$
$$= 1.5 \times 10^{-4}\,\text{A} = 150\,\mu\text{A}$$

Current sensitivity of galvanometer is given by

$$= \frac{10\,\text{div}}{150\,\mu\text{A}} = \frac{1}{15}\,\text{div}/\mu\text{A}$$
$$\approx 0.066\,\text{div}/\mu\text{A}$$

**13.** (d)




Given, $\quad P = 50\,\Omega, Q = 100\,\Omega$

**Case I** For the balancing point,

$$\frac{R_{AC}}{P} = \frac{R_{CB}}{Q} \Rightarrow \frac{R_{AC}}{50} = \frac{R_{CB}}{100}$$
$$\Rightarrow \quad 2R_{AC} = R_{CB}$$
$$\Rightarrow \quad \frac{2\rho(33 - a)}{A} = \rho\left(\frac{b - 33}{A}\right)$$
$$\left[\because R = \frac{\rho l}{A} \text{ and given } AC = 33\,\text{cm}\right]$$
$$\Rightarrow \quad 2(33 - a) = (b - 33)$$
$$\Rightarrow \quad 66 - 2a = b - 33 \quad \ldots\text{(i)}$$

**Case II** When resistors are interchanged

and $\quad AC = 67\,\text{cm}$

$$\frac{R_{AC}}{100} = \frac{R_{CB}}{50}$$
$$\Rightarrow \quad R_{AC} = 2R_{CB}$$
$$\Rightarrow \quad \frac{\rho(67 - a)}{A} = 2\rho\left(\frac{b - 67}{A}\right)$$
$$\Rightarrow \quad 67 - a = 2b - 134 \quad \ldots\text{(ii)}$$

After multiplying Eq. (ii) with 2, we get

$$134 - 2a = 4b - 268 \quad \ldots\text{(iii)}$$

Subtracting Eq. (i) from Eq. (iii), we get

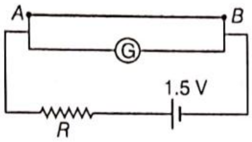
$$134 - 2a = 4b - 268$$
$$66 - 2a = b - 33$$
$$\underline{-\quad + \quad - \quad +}$$
$$68 = 3b - 235$$
$$3b = 303 \Rightarrow b = 101\,\text{cm}$$

From Eq. (i),

$$66 - 2a = 101 - 33 = 68$$
$$2a = -2 \Rightarrow a = -1\,\text{cm}$$

So, end corrections are 1 and 1.

**14.** (c)




**Case I** Current through $G$

$$I_0 = \frac{E}{R+G}$$

**Case II** Current through $G$

$$I_1 = \frac{E}{\left(\frac{GS}{S+G} + R\right)} \times \frac{S}{S+G}$$

$S$ = resistance

Above relation is true, if $R >> G$

**15.** (d) Given, dimensions of $A = [\text{LT}]$

Dimensions of $B = [\text{L}^2\text{T}^{-1}]$

Dimensions of $C = [\text{LT}^2]$

$$A = B^nC^m$$
$$[\text{LT}] = [\text{L}^2\text{T}^{-1}]^n [\text{LT}^2]^m = \text{L}^{2n+m}\,\text{T}^{2m-n}$$

After comparing, we get

$$m + 2n = 1 \quad \ldots\text{(i)}$$
$$2m - n = 1 \quad \ldots\text{(ii)}$$

Multiplying Eq. (ii) with 2, we get

$$4m - 2n = 2 \quad \ldots\text{(iii)}$$

Adding Eqs. (i) and (iii), we get

$$m + 2n + 4m - 2n = 1 + 2$$
$$5m = 3 \Rightarrow m = \frac{3}{5}$$

From Eq. (ii), $2m - n = 1$

$$n = 2m - 1 = 2 \times \frac{3}{5} - 1 = \frac{6-5}{5} = \frac{1}{5}$$

**16.** (b) In both rooms, pressure is same and they have same volume.

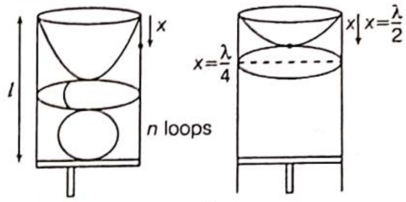
So, $pV = nRT = \text{constant} = n \cdot T = \text{constant}$

$$T \propto \frac{1}{n}$$

**17.** (b) Given, distance moved by piston = 8.75 cm

Speed of sound in air ($v$) = 350 m/s

According to question, we can draw the following diagram,




Now, displacement $= \frac{\lambda}{4} = 8.75$

$$\lambda = 35\,\text{cm or } \frac{35}{100}\,\text{m}$$

We know that, $f = \frac{v}{\lambda} = \frac{350 \times 100}{35} = 1000\,\text{Hz}$

**18.** (a) Given, speed of heavy metal block = 20 km/h

Coefficient of friction between the block and surface = 0.6

Specific heat of block material = 0.1 Cal/g-°C

We know that,

Distance = velocity × time

Here, time ($t$) = 10 min

$$\text{Distance} = \frac{20}{60} \times 10 = \frac{10}{3}\,\text{km} = \frac{10000}{3}\,\text{m}$$

Work done by friction,

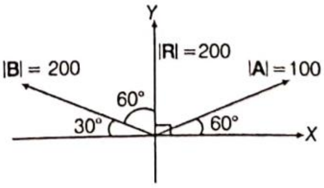
$$W = \mu\, mg \cdot s$$
$$W = 0.6 \times m \times 10 \times \frac{1}{3} \times 10^4 = 2m \times 10^4\ \text{J}$$

According to question, 25% of work done converts into heat 50

$$\frac{1}{4}W = msdt$$
$$\Rightarrow \quad \frac{1}{4} \times (2m \times 10^4) = m \times 0.1 \times 42 \times dt \times 10^3$$
$$dt = \frac{20}{4 \times 0.1 \times 42} = \frac{5}{0.42} = 11.9°\text{C} \approx 12°\text{C}$$

**19.** (b) Internal energy change in all paths is same so work done in all paths should be same.

**20.** (d) According to question, we can draw the following diagram,



From figure,

$$\mathbf{A} = 100\cos 60^\circ\,\hat{i} + 100\sin 30^\circ\,\hat{j}$$
$$\mathbf{A} = 100 \times \frac{1}{2}\,\hat{i} + 100 \times \frac{\sqrt{3}}{2}\,\hat{j}$$
$$\mathbf{A} = 50\,\hat{i} + 50\sqrt{3}\,\hat{j}$$

Similarly, $\mathbf{B} = -100\sqrt{3}\,\hat{i} + 100\,\hat{j}$

$$\mathbf{R} = 200\hat{j}$$

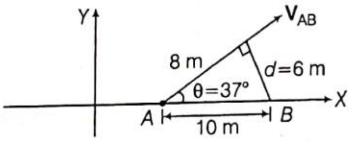
Now, $\mathbf{R} = \mathbf{A} + \mathbf{B} + \mathbf{C}$

$$\mathbf{C} = \mathbf{R} - \mathbf{A} - \mathbf{B}$$
$$\mathbf{C} = 200\,\hat{j} - 50\,\hat{i} - 50\sqrt{3}\,\hat{j} + 100\sqrt{3}\,\hat{i} - 100\hat{j}$$
$$\mathbf{C} = (100\sqrt{3} - 50)\hat{i} + (100 - 50\sqrt{3})\,\hat{j}$$
$$\mathbf{C} \approx 120\,\hat{i} + 15\,\hat{j}$$
$$|\mathbf{C}| = \sqrt{(100\sqrt{3} - 50)^2 + (100 - 50\sqrt{3})^2}$$
$$= \sqrt{(100)^2\,(4) + (50)^2\,4 - 4 \times 50 \times 100 \times \sqrt{3}}$$
$$= 2(50)\sqrt{4 + 1 - 2\sqrt{3}}$$
$$= 100\sqrt{5 - 2\sqrt{3}} = 100\sqrt{5 - 3\cdot 4}$$
$$= 100 \times 1239 = 124 \text{ units}$$
$$\tan\theta = \frac{15}{120}$$
$$\theta = \tan^{-1}(0.125) \approx 6.2^\circ$$

**21.** (b) Given, the distance between particles $A$ and $B = 10\,\text{m}$

Velocity of particle $A$, $(v_A) = 0.75\,\text{m/s}$

Velocity of particle $B$, $(v_B) = 1\,\text{m/s}$




$$\mathbf{V}_{AB} = \mathbf{V}_A - \mathbf{V}_B = 0.75\,\hat{j} + \hat{i}$$
$$\mathbf{V}_{AB} = \hat{i} + \frac{3}{4}\,\hat{j}$$
$$\tan\theta = \frac{3}{4}$$
$$\theta = \tan^{-1}\left(\frac{3}{4}\right) = 37^\circ$$
$$t = \frac{10\cos\theta}{\sqrt{1 + \frac{9}{16}}}$$
$$= \frac{10 \times \frac{4}{5}}{\sqrt{1 + \frac{9}{16}}} = \frac{10 \times 4/5}{5/4} = \frac{160}{25}$$
$$= 6.4\,\text{s}$$

**22.** (d) We know that,

The time period of second pendulum is $T = 2$ s

$$\frac{2\pi}{\omega} = 2$$
$$\omega = \pi \quad \text{...(i)}$$

So, equation of motion is

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

From Eq. (i), $\frac{d^2x}{dt^2} = -\pi^2 x$

$$\Rightarrow \quad \frac{d^2x}{dt^2} + \pi^2 x = 0$$

**23.** (d) Given, mass of block $(m) = 2$ kg

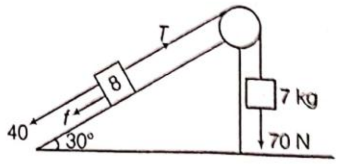
Vertical height $(h) = 0.4$ m

Spring constant $(K) = 1960$ N/m

According to question,

$$mg\,(h + x) = \frac{1}{2}Kx^2$$
$$\Rightarrow \quad 2 \times 9.8\,(0.4 + x) = \frac{1}{2}(1960)\,x^2$$
$$\Rightarrow \quad 2(0.4 + x) = 100\,x^2$$
$$\Rightarrow 100\,x^2 - 2x - 0.8 = 0$$
$$x = \frac{-b \pm \sqrt{b^2 + 4ac}}{2a}$$
$$x = \frac{2 \pm \sqrt{4 + 320}}{200}$$
$$x = \frac{2 + \sqrt{324}}{200} = \frac{2 + 18}{200} = \frac{20}{200}$$
$$= 0.1\,\text{m}$$

**24.** (b) According to question, we can draw the following diagram,




From figure, $T = 70$ N and $40 + f = T$

$$40 + f = 70$$
$$f = 70 - 40 = 30\,\text{N down the plane}$$

**25.** (b) Given, frequency of sirens $= 400$ Hz

Speed of man $(v_0) = 2$ m/s

Speed of sound $(v) = 320$ m/s

Frequency heard from first factory,

$$f_1 = f_0\left(\frac{v + v_0}{v}\right) \quad \text{...(i)}$$

Frequency heard from second factory,

$$f_2 = f_0\left(\frac{v - v_0}{v}\right) \quad \ldots\text{(ii)}$$

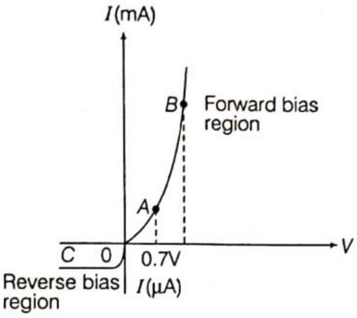
Beat frequency, $\Delta f = f_1 - f_2$

From Eqs. (i) and (ii),

$$\Delta f = f_0\left(\frac{v + v_0}{v}\right) - f_0\left(\frac{v - v_0}{v}\right)$$

$$\Delta f = f_0\left(\frac{2v_0}{v}\right) = 400\left(\frac{2 \times 2}{320}\right) = \frac{160}{32} = 5 \text{ beat/s}$$

**26.** (c) *I-V* characteristics of silicon diode is




So, statements I and III are correct.

**27.** (a) Given, refractive index of lens materials

$(\mu_L) = 1.5$

Focal length of lens $(F) = 12$ cm

Refractive index of liquid $(\mu_m) = 1.35$

Now, $F_{\text{air}} = \frac{F}{2} = \frac{12}{2} = 6$ cm

Focal length in the medium is $f_{\text{medium}}$, then

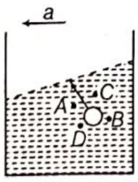
$$\frac{F_m}{F_{\text{air}}} = \frac{\mu_L - 1}{\left(\frac{\mu_L}{\mu_m} - 1\right)} = \frac{1.5 - 1}{\frac{1.5}{1.35} - 1}$$

$$\frac{F_m}{6} = \frac{0.5}{\frac{0.15}{1.35}}$$

$$F_m = \frac{6 \times 0.5 \times 1.35}{0.15} = 27 \text{ cm}$$

Focal length is positive, so it acts as convex lens.

**28.** (d) If car is accelerating towards left, then

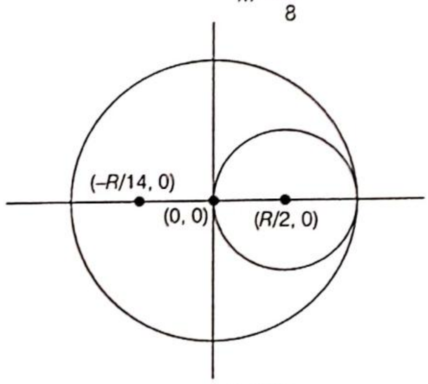



$p_B > p_A$ and $p_D > p_C$

So, net force would be inclined in the left.

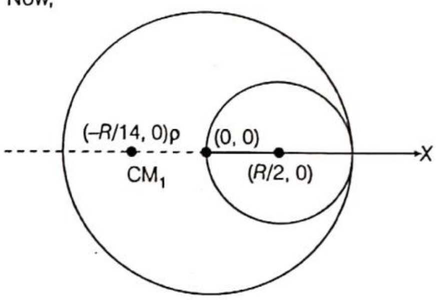
**29.** (b) CM of the remaining styrofoam sphere is

$$X'_{CM} = \frac{M(0) - \frac{M}{8}\left(\frac{R}{2}\right)}{M - \frac{M}{8}}$$




$$X'_{CM} = \frac{-\frac{M}{8}\left(\frac{R}{2}\right)}{\frac{7M}{8}} = \frac{-R}{14}$$

Now,

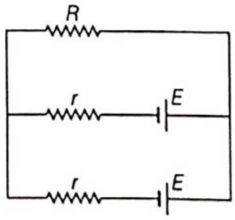



$$\text{Mass, } M' = 5\rho\left(\frac{4}{3}\pi \cdot \frac{R^3}{8}\right) = \frac{5}{8}\left(\rho \cdot \frac{4}{3}\pi R^3\right) = \frac{5M}{8}$$

$$X_{CM} = \frac{\frac{7M}{8}\left(\frac{-R}{14}\right) + \frac{5M}{8}\left(\frac{R}{2}\right)}{\frac{7M}{8} + \frac{5M}{8}}$$

$$X_{CM} = \frac{\frac{4MR}{8}}{\frac{12M}{R}} = \frac{R}{3} \text{ from the centre}$$

**30.** (a)

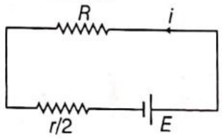


From the circuit diagram,

$$E_{net} = \frac{\frac{2E}{r}}{\frac{2}{r}} = E$$

Similarly, $r_{eq} = \frac{r}{2}$

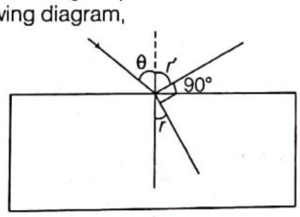
Now, circuit becomes




$$i = \frac{E}{R + \frac{r}{2}}$$

$$V_R = \left(\frac{E}{R + \frac{r}{2}}\right) \times R = \frac{2ER}{2R + r}$$

**31.** (b) According to question, we can draw the following diagram,




From figure,

$$\theta = r' \quad \ldots(i)$$

$$r' + r + 90° = 180°$$

$$r + r' = 90°$$

$$r = 90° - r' \quad \ldots(ii)$$

According to Snell law,

$$\mu = \frac{\sin\theta}{\sin r}$$

$$\sin\theta = \mu \sin r$$

$$\sin\theta = \mu \sin(90° - r') \quad \text{[from Eq. (ii)]}$$

$$\sin\theta = \mu \cos r'$$

$$\sin\theta = \mu \cos\theta \quad \text{[from Eq. (i)]}$$

$$\mu = \tan\theta$$

$$90° - r \sin\phi = \frac{1}{\tan\theta}$$

$$\sin\phi = \frac{1}{\tan(90° - r)}$$

$$\sin\phi = \frac{1}{\cot r} \Rightarrow \sin\phi = \tan r$$

$$\Rightarrow \quad r = \tan^{-1}(\sin\phi)$$

**32.** (c) Percentage decrease of radius = 0.1%

We know that,

$$R = \rho\frac{l}{A}$$

$$R = \rho\frac{V}{A^2} \quad \left[\because l = \frac{V}{A}\right]$$

$$R = \frac{\rho V}{(\pi r^2)^2} \quad [\because A = \pi r^2]$$

$$R = \frac{\rho V}{\pi^2 r^4}$$

Here, ρ, $V$ and π are constant, then

$$\frac{\Delta R}{R} = 4\frac{\Delta r}{r}$$

$$\left(\frac{\Delta R}{R} \times 100\right) = 4\left(\frac{\Delta r}{r} \times 100\right)$$

$$\frac{\Delta R}{R} \times 100 = 4 \times 0.1$$

$$\frac{\Delta R}{R} \times 100 = 0.4\%$$

Hence, the percentage changes in resistance is 0.4%.

**33.** (a) Given, focal length of lens ($f$) = 5 cm

Distance of object ($u$) = – 30 cm

We know that,

$$\frac{1}{f} = \frac{1}{v} - \frac{1}{u}$$

$$\Rightarrow \quad \frac{1}{5} = \frac{1}{v} - \frac{1}{(-30)}$$

$$\Rightarrow \quad \frac{1}{5} = \frac{1}{v} + \frac{1}{30}$$

$$\Rightarrow \quad \frac{1}{v} = \frac{1}{5} - \frac{1}{30}$$

$$\Rightarrow \quad \frac{1}{v} = \frac{6-1}{30} = \frac{5}{30}$$

$$\Rightarrow \quad v = \frac{30}{5}$$

$$v = 6\,\text{cm}$$

Thus, lens has to be moved out by 1cm.

**34.** (d) We know that,

$$\text{Fringe width } (\beta) = \frac{\lambda D}{d}$$

As the air is evacuated, the wavelength increases and hence fringe width also increases.

**35.** (b) Changing the amplitude will not change the locations of increased and decreased sound intensity.

**36.** (c) Given, the mass of particles are $m_1$ and $m_2$

and $m_1 > m_2$

de-Broglie wavelength of $m_1$

$$\lambda_1 = \frac{h}{m_1 v_1} \quad \ldots(i)$$

de-Broglie wavelength of $m_2$

$$\lambda_2 = \frac{h}{m_2 v_2} \quad \ldots(ii)$$

Momentum is conserved so momentum of each particle is equal in magnitude. i.e.,

$$m_1 v_1 = m_2 v_2$$

From Eqs. (i) and (ii), we get

$$\lambda_1 = \lambda_2 \quad [\because h = \text{Planck constant}]$$

**37.** (b) Given, masses of particles are $m_1$ and $m_2$ and radii of circular path are $R_1$ and $R_2$.

For first circular path

Radius, $$R_1 = \frac{m_1 v_1}{qB} \quad \ldots(i)$$

But $$\frac{1}{2} m_1 v_1^2 = qv_0$$

$$\frac{1}{2m_1} m_1^2 v_1^2 = qv_0$$

$$(m_1 v_1)^2 = 2m_1 qv_0$$

$$m_1 v_1 = \sqrt{2m_1 qv_0} \quad \ldots(ii)$$

On substituting the value from Eq. (ii) into Eq. (i), we get

$$R_1 = \frac{\sqrt{2m_1 qv_0}}{qB} \quad \ldots(iii)$$

Similarly, $$R_2 = \frac{\sqrt{2m_2 qv_0}}{qB} \quad \ldots(iv)$$

From Eqs. (iii) and (iv), we get $\left(\frac{m_1}{m_2}\right) = \left(\frac{R_1}{R_2}\right)^2$

**38.** (c) Given, current in wire $(M)$, $I_1 = 200\,\text{A}$

Current in wire $(N)$, $I_2 = I$

Distance $(d) = 20\,\text{mm} = 20 \times 10^{-3}\,\text{m}$

Linear mass density of wire $(\lambda) = 10^{-2}\,\text{kg/m}$

We know that,

$$\lambda g = \frac{\mu_0 I_1 I_2}{2\pi d}$$

$$10^{-2} \times 9.8 = \frac{2 \times 10^{-7} \times 200 \times I}{20 \times 10^{-3}} \left[\because \frac{\mu_0}{2\pi} = 2 \times 10^{7}\right]$$

$$I = 10^2 \times \frac{9.8}{20} = \frac{98}{20} \approx 49\,\text{A}$$

**39.** (a) Given, intensity of unpolarised light

$$(I_0) = 32\,\text{W/m}^2$$

Intensity of emergent light $(I_3) = 3\,\text{W/m}^2$

According to question, the intensity of unpolarised light becomes half after passing first polarizer i.e.,

$$I_1 = 16\,\text{W/m}^2 \quad \ldots(i)$$

After passing second polariser, intensity of light,

$$I_2 = 16\cos^2\theta \quad \ldots(ii)$$

After passing third polariser, intensity of light,

$$I_3 = (16\cos^2\theta)\sin^2\theta$$

$$\Rightarrow \quad 3 = 16\cos^2\theta\sin^2\theta$$

$$\Rightarrow \quad 4 \times 4\cos^2\theta\sin^2\theta = 3$$

$$\Rightarrow \quad (2\cos\theta\sin\theta)^2 = 3/4$$

$$\Rightarrow \quad (\sin 2\theta)^2 = \frac{3}{4}$$

$$\Rightarrow \quad \sin 2\theta = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad 2\theta = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = 60^\circ$$

$$\Rightarrow \quad \theta = \frac{60^\circ}{2} = 30^\circ$$

So, the angle between the transmission axes of the first two polarisers is 30°.

**40.** (a) Given, surface charge density of metal plate

$$(\sigma) = -2.0 \times 10^{-6}$$

Initial KE of electron $= 1.6 \times 10^{-17}\,\text{J}$

According to question, by conservation of energy,

Loss in kinetic energy = gain in potential energy

According to question, loss in kinetic energy is equal to kinetic energy. So,

$$\text{KE} = (eE)d$$

$$\text{KE} = e\left(\frac{\sigma}{\varepsilon_0}\right)d \quad \left[\because E = \frac{\sigma}{\varepsilon_0}\right]$$

$$\Rightarrow \quad d = \frac{\text{KE} \times \varepsilon_0}{e \times \sigma}$$

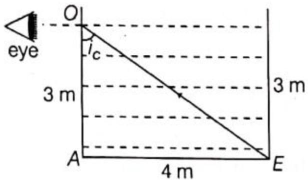
$$= \frac{1.6 \times 10^{-17} \times 8.85 \times 10^{-12}}{1.6 \times 10^{-19} \times 2.0 \times 10^{-6}}$$

$$= \frac{8.85}{2.0} \times 10^{-4}$$

$$= 4.4 \times 10^{-4}\,\text{m}$$

**41.** (d) The more is the area enclosed, the more is the work done by the gas. Thus, in option (d) the gas performs greatest amount of work.

**42.** (d) According to question, we can draw following diagram,




In $\Delta AOE$,

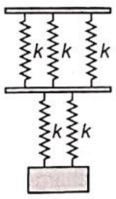
$$OE^2 = AE^2 + AO^2$$
$$OE^2 = (4)^2 + (3)^2$$
$$OE = \sqrt{16+9} = \sqrt{25} = 5\,\text{m}$$

Now, $\sin i_c = \dfrac{4}{5}$

We know that,

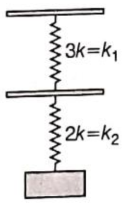
$$\mu = \frac{1}{\sin i_c} = \frac{5}{4} = 1.25$$

**43.** (d)




Here, $k$ = spring constant

Springs are connected in parallel combination so, we can draw the above diagram in following manner.




Now, springs are connected in series combination.

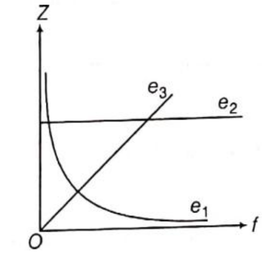
So, equivalent spring constant,

$$(k_{eq}) = \frac{k_1k_2}{k_1+k_2} = \frac{(3k)(2k)}{3k+2k}$$
$$= \frac{6k^2}{5k} = \frac{6}{5}k$$

Frequency of oscillation of the block

$$\nu = \frac{1}{2\pi}\sqrt{\frac{k_{eq}}{m}} = \frac{1}{2\pi}\sqrt{\frac{6k}{5m}}$$

**44.** (d)




From diagram,

Since, $X_C = \dfrac{1}{\omega_C}$. So, $e_1$ represents capacitor $C$.

Since, $X_L = \omega L$. So, $e_3$ represents inductor $L$.

Since, $R$ is a constants. So, $e_2$ represents resistor $R$.

**45.** (a) According to question,

Half-life of radioactive element $E_1$ = mean life of radioactive element $E_2$

$$\frac{\ln 2}{\lambda_1} = \frac{1}{\lambda_2}$$

Here, $\lambda_1$ and $\lambda_2$ are the decay constants of $E_1$ and $E_2$.

$$\Rightarrow \quad \frac{0.693}{\lambda_1} = \frac{1}{\lambda_2}$$

$$\therefore \quad \lambda_2 > \lambda_1$$

Decay constant of $E_2$ is greater than $E_1$.

Thus, $E_2$ decays faster.

**46.** (b) We know that,

Time period of physical pendulum is given by

$$T = 2\pi\sqrt{\frac{I}{MgL}} \quad \ldots\text{(i)}$$

Moment of inertia of rod,

$$I = \frac{ml^2}{3} + ml^2 = \frac{4ml^2}{3} \quad \ldots\text{(ii)}$$

$$L = \frac{m\left(\frac{l}{2}\right) + m(l)}{2m} = \frac{3l}{4} \quad \ldots\text{(iii)}$$

On substituting the value from Eqs. (ii) and (iii) in Eq. (i), we get

$$T = 2\pi\sqrt{\frac{\frac{4ml^2}{3}}{(2mg)\frac{3l}{4}}}$$

$$T = 2\pi \times \frac{4}{3}\sqrt{\frac{l}{2g}} = 2\pi\sqrt{\frac{8}{9}\cdot\frac{l}{g}}$$

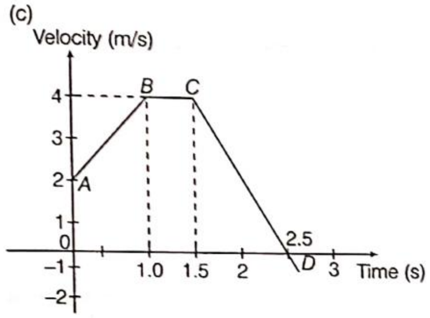
**47.** (b) The upward maximum velocity of the ball is represented by point '$B$' the one just after the first collision.

**48.** (d) We know that,

Frequency $(f) = \frac{1}{2\pi}\sqrt{\frac{2k}{m}}$

Thus, increasing '$k$' will increase frequency. Thus, using harder pairs of springs will serve the purpose.

**49.** (c)




From graph,

For $AB$, $a_1$ = slope of $AB$

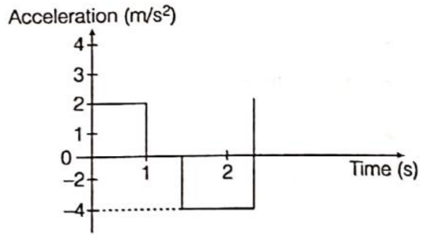
$$a_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-2}{1-0} = \frac{2}{1} = 2\,\text{m/s}^2$$

For $BC$, $a_2 = 0$ [$\because$ velocity is constant]

For $CD$, $a_3$ = slope of $CD$

$$= \frac{y_2 - y_1}{x_2 - x_1} = \frac{0-4}{2.5-1.5} = \frac{-4}{1.0} = -4\,\text{m/s}^2$$

Now, the acceleration *versus* time graph will be




**50.** (c) A DC voltmeter reads an average of full cycle. Thus, the value must have been zero but as there is a rectifier thus, it will give reaching of average of half-cycle.

**51.** (d) We know that

Intensity of wave $(I) \propto$ [Frequency $(f)]^2$

So,

$$\frac{I_2}{I_1} = \frac{(1200)^2}{(400)^2} = \frac{(3)^2}{(1)^2} = \frac{9}{1}$$

**52.** (c) A bridge rectifier is a full wave rectifier. So, frequency gets doubled i.e., $f = 2 \times 50 = 100\,\text{Hz}$

**53.** (c) Given, $f = \frac{ke^2}{r^2}$

We know that,

$$\frac{ke^2}{r^2} = mv^2$$

$$\text{KE} = \frac{ke^2}{2r} + \left(-\frac{ke^2}{r}\right) = \frac{ke^2}{2r}$$

For initial stage,

$$mv^2 = -\frac{ke^2}{R_1}$$

$$\text{KE}_i = mv^2 = -\frac{ke^2}{2R_1} \quad \text{...(i)}$$

Similarly, for final image,

$$\frac{1}{2}mv^2 = -\frac{ke^2}{2R_2}$$

$$\text{KE}_f = -\frac{ke^2}{2R_2} \quad \text{...(ii)}$$

Now, $\Delta\text{KE} = (\text{KE})_i - (\text{KE})_f$

$$= -\frac{ke^2}{2R_1} - \left(\frac{-ke^2}{2R_2}\right)$$

$$= \frac{ke^2}{2}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)$$

**54.** (b) We know that,

$$\frac{1}{\lambda} = Rd\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)Z^2$$

$$\therefore \left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right) = \left(\frac{1}{m_1^2} - \frac{1}{m_2^2}\right)$$

$$= \left(\frac{1}{6^2} - \frac{1}{2^2}\right) = \frac{1}{m_1^2} - \frac{1}{m_2^2}$$

$$\Rightarrow \frac{1}{m_1^2} - \frac{1}{m_2^2} = \left(\frac{1}{3^2} - \frac{1}{2^2}\right)$$

Thus, a transition from 3 to 2 in H-atom is equivalent to transition from 6 to 4 in $He^+$ ion.

**55.** (a) Excess pressure inside the tyre increases the inflation of tyre. The horizontal road surface is not exactly tangent to the road along a line.

**56.** (c) According to question,

$$4 \times L \times W \times (\text{pressure}) = 1500$$

$$\Rightarrow 4 \times L \times 16.5 \times 10^{-2} \times 2 \times 10^4 = 1500$$

$$\Rightarrow L = \frac{11.36}{100}\,\text{m}$$

$$\Rightarrow L = 11.36\,\text{cm}$$

**57.** (a) According to question,

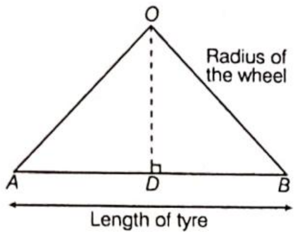
$$\Rightarrow 4\times(L+2.5)\times10^{-2}\times16.5\times10^{-2}\times2\times10^4 = (1500+5x)$$

$$\Rightarrow 4\times(11.36\times2.5)\times10^{-2}\times16.5\times10^{-2}\times2\times10^4 = 1500+5x$$

After solving, we get

$$x = 65.9\,\text{kg} \approx 66\text{ kg}$$

**58.** (c) According to question, we can draw the following diagram,




Given, $AB = 13.86\,\text{cm}$

$BD = 6.9\,\text{cm}$

$OB = 27.7\text{ cm}$

In $\Delta ODB$, $OD = \sqrt{OB^2 - BD^2}$

$= \sqrt{(27.7)^2 - (6.93)^2}$

$= 26.8\,\text{cm}$

Again, $AB = 11.36\,\text{cm}$

$BD = 5.66\,\text{cm}$

$OB = 27.7\text{ cm}$

In $\Delta ODB$, $OD^2 = \sqrt{OB^2 - BD^2}$

$OD = \sqrt{(27.7)^2 - (5.66)^2}$

$OD = 27.1\,\text{cm}$

Reduced height $= 27.1 - 26.8$

$= 0.3\,\text{cm} = 3\,\text{mm}$

**59.** (b) Volume $= \pi\,[(27.7)^2 - (20)^2]\,16.5$

$= 19.029\,\text{L}$

Mass of air $=$ Volume $\times\rho\times2$

$= 19.029\times2\times129$

$= 49.095\,\text{g}$

**60.** (a) For increasing stability and acceleration, the width of tyre is large.

**61.** (b,c) Given, rate of flow = 52.5 mL/s

Speed of flow $(v_1) = 0.35\,\text{m/s}$

Area of cross-section $(A_2) = 0.5\,\text{cm}^2$

From equation of continuity,

$$A_1v_1 = A_2v_2 \quad \ldots\text{(i)}$$

Rate of flow, $A_1v_1 = 52.5$

$$A_1 = \frac{52.5}{35}\,\text{cm}^2 = 1.5\,\text{cm}^2$$

From Eq. (i),

$$v_2 = \frac{A_1v_1}{A_2}$$

$$v_2 = \frac{1.5}{0.5}\times0.35 = 1.05\,\text{m/s}$$

According to Bernoulli theory,

$$\frac{1}{2}\rho v_1^2 + \rho gh = \frac{1}{2}\rho v_2^2$$

$$\Rightarrow \quad v_2^2 - v_1^2 = 2gh$$

$$\Rightarrow \quad h = \frac{v_2^2 - v_1^2}{2g}$$

$$= \frac{(1.05)^2 - (0.35)^2}{2\times9.8}$$

$$= \frac{1.10 - 0.12}{2\times9.8} = \frac{0.98}{2\times9.8} = 0.05\text{ m}$$

**62.** (b,c) Since, efficiency of half-wave is less than full wave rectifier, output power decreased and since energy is only passing through, one path and hence, ripple increases.

**63.** (a,c) Given, $C_A = C,\ L_A = L$ [for circuit $A$]

$C_B = 2C,\ L_B = \frac{L}{2}$ [for circuit $B$]

We know that the resonance frequency,

$$f = \frac{1}{2\pi\sqrt{LC}}$$

For circuit $A$,

Resonance frequency $(f_A) = \dfrac{1}{2\pi\sqrt{LC}}$ ...(i)

For circuit $B$,

Resonance frequency $(f_B) = \dfrac{1}{2\pi\sqrt{\frac{L}{2}\times2C}}$

$$f_B = \frac{1}{2\pi\sqrt{LC}}$$

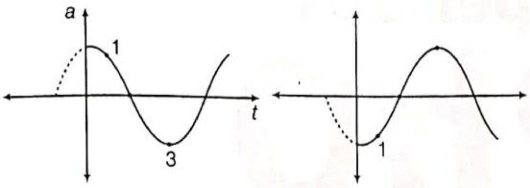
$f_B = f_A$ [from Eq. (i)]

Peak current $= \dfrac{e_0}{R}$

$$\theta = \frac{\omega L}{R}$$

$$\Rightarrow \quad \theta \propto L$$

**64.** (a,b,c,d)




$$a = -\omega^2 x$$

$\therefore$ $t = 0$, $x$ = non-zero negative value

At point 2, $v = \dfrac{dx}{dt} > 0$

At point 3, velocity = 0

Hence, potential energy is maximum.

**65.** (a,b,c) $\varepsilon_0 E^2$ and $\dfrac{B^2}{\mu_0}$ have dimensions same as energy per volume.

$Y$ has dimensions of pressure $= \dfrac{F}{A} = \dfrac{\text{Energy}}{\text{Volume}}$

$KT$ has dimensions of energy/mole.

**66.** (b,d) Speed at any instant to travel the first half of the distance is less than terminal velocity.

$\therefore$ $t_1 > t_2$

Work done against the viscous drag while travelling first half of the distance

$$-W_1 + mgh = \frac{1}{2}mv^2 \Rightarrow -W_2 + mgh = 0$$

$\therefore$ $W_1 < W_2$

**67.** (a,b,c,d) Given, $x = kt$ ...(i)

$$y = kt\,(1 - pt) \quad \text{...(ii)}$$

From Eq. (i), $t = \dfrac{x}{k}$ ...(iii)

On substituting the value from Eq. (iii) in Eq. (ii), we get

$$y = k \times \frac{x}{k}\left(1 - p\frac{x}{k}\right)$$

After solving, we get

$$y = x - px^2$$

$\therefore$ The trajectory is a parabola.

$$v = \frac{dx}{dt} = k$$

Since, $x = kt$

$\therefore$ Velocity of the particle along $X$-axis is constant.

$$a_y = -kp < 0$$

$\because$ $k$ and $p$ are both negative, $a_y < 0$

Also, $a_y$ is along $-Y$-axis.

**68.** (a,b,c) According to question,

$$R_A = 2\hat{i} - 3\hat{j} - \hat{k}$$
$$R_B = -\hat{i} - 2\hat{j} + 4\hat{k}$$
$$R_C = 2\hat{i} - 4\hat{j} + \hat{k}$$
$$|R_A| = \sqrt{(2)^2 + (-3)^2 + (-1)^2} = \sqrt{4 + 9 + 1} = \sqrt{14}$$

Similarly, $|R_B| = \sqrt{(-1)^2 + (-2)^2 + (4)^2} = \sqrt{21}$

$$|R_C| = \sqrt{(2)^2 + (-4)^2 + (1)^2} = \sqrt{21}$$
$$R_B = R_C \Rightarrow V_B = V_C$$

No work is done to take charge from $B$ to $C$. $\mathbf{E}_A$ is in the direction of $\mathbf{R}_A$ and $\mathbf{E}_B$ is in the direction of $\mathbf{R}_B$.

$$\mathbf{R}_A \cdot \mathbf{R}_B = (2\hat{i} - 3\hat{j} - \hat{k}) \cdot (-\hat{i} - 2\hat{j} + 4\hat{k})$$
$$= -2 + 6 - 4 = 0$$

$\therefore$ $\mathbf{E}_A \cdot \mathbf{E}_B = 0$

$\Rightarrow$ $\mathbf{E}_A \perp \mathbf{E}_B$

We know that,

$$E = \frac{kQ}{R^2}$$
$$E \propto \frac{1}{R^2}$$

Hence, $\dfrac{E_A}{E_B} = \dfrac{R_B^2}{R_A^2} = \dfrac{21}{14} = \dfrac{3}{2}$

$\Rightarrow$ $2E_A = 3E_B$

**69.** (c,d) If $r_1$ and $r_2$ are both less than $R$, then $E \propto r$.

or $\dfrac{E_1}{E_2} = \dfrac{r_1}{r_2}$

If $r_1 > R$ and $r_2 > R$, then

$$\frac{E_1}{E_2} = \frac{r_2^2}{r_1^2}$$

If $r_1 > R$ and $r_2 < R$, then

$$\frac{E_1}{E_2} = \frac{R^3}{r_2 r_1^2}$$

**70.** (a,b,c,d) We know that,

Tension $(T) = \dfrac{YA\Delta x}{l} = \dfrac{YA\alpha\theta}{l} = YA\alpha\theta$ ...(i)

$$v = \sqrt{\frac{T}{\mu}}$$

From Eq. (i), $v = \sqrt{\dfrac{YA\alpha\theta}{\mu}} = \sqrt{\dfrac{Y\alpha\theta}{\rho}}$ ...(ii)

$$f = \frac{v}{2l}$$

From Eq. (ii), $f = \dfrac{1}{2l}\sqrt{\dfrac{Y\alpha\theta}{\rho}}$

# Solved Paper 2017

# INPhO

## Indian National Physics Olympiad

*Conducted by: Homi Bhabha Centre for Science Education, India*

**Exam Held on** 29-01-2017

*Students selected from Stage I examination (NSEP) are eligible to appear for INPhO. On the basis of performance in INPhO, the top 35 students in the merit list are selected and out of these 35 students finally 5 students go to participate in International Physics Olympiad.*

**Table of Constants**

| | | |
|---|---|---|
| Speed of light in vacuum | $c$ | $3.00 \times 10^8 \text{ms}^{-1}$ |
| Planck's constant | $h$ | $6.63 \times 10^{-34}$ Js |
| Universal constant of Gravitation | $G$ | $6.67 \times 10^{-11} \text{Nm}^2\text{kg}^{-2}$ |
| Magnitude of electron charge | $e$ | $1.60 \times 10^{-19}$C |
| Mass of electron | $m_e$ | $9.11 \times 10^{-31}$kg |
| Value of $\frac{1}{4\pi \epsilon_0}$ | | $9.00 \times 10^9 \text{Nm}^2\text{C}^{-2}$ |
| Universal Gas Constant | $R$ | $8.31 \text{JK}^{-1} \text{mole}^{-1}$ |

**1.** A massive star of mass $M$ is in uniform circular orbit around a supermassive black hole of mass $M_b$. Initially, the radius and angular frequency of the orbit are $R$ and $\omega$ respectively. According to Einstein's theory of general relativity the space around the two objects is distorted and gravitational waves are radiated. Energy is lost through this radiation and as a result the orbit of the star shrinks gradually. One may assume, however, that the orbit remains circular throughout and Newtonian mechanics holds.

(a) The power radiated through gravitational wave by this star is given by **[1]**

$$L_G = Kc^x G^y M^2 R^4 \omega^6$$

where $c$ is the speed of light, $G$ is the universal gravitational constant and $K$ is a dimensionless constant. Obtain $x$ and $y$ by dimensional analysis.

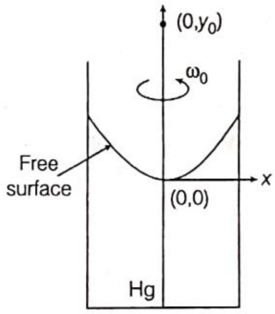
(b) Obtain the total mechanical energy ($E$) of the star in terms of $M$, $M_b$ and $R$. **[1]**

(c) Derive an expression for the rate of decrease in the orbital period $(dT/dt)$ in terms of the masses, period $T$ and constants. **[3]**

**2.** The free surface of mercury (Hg) is a good reflecting surface. A tall cylinder partly filled with Hg and possessing total moment of inertia $I$ is rotated about its axis with the constant angular velocity $\omega_0$ as shown in figure. The Hg surface attains a paraboloidal profile. The radius of curvature $\rho$ of the Hg surface attains a paraboloidal profile. The radius of curvature $\rho$ of a general profile is given by

$$\rho = \left| \frac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2} \right|$$

Where the symbols have their usual meaning.




(a) Obtain the expression for $\rho$ of the Hg surface in terms of $\omega_0$, the distance $x$ from the cylinder axis and $g$. **[3]**

(b) Calculate the value of $\rho$ at the lowest point of the Hg surface, that is (0, 0), when $\omega_0 = 78$ rpm (revolution per minute) **[1]**

(c) Consider a point object at $(0, y_0)$ as shown in the figure. Obtain an expression for the image position $y_i$ in terms of given quantities. State conditions on $y_0$ for the formation of real and virtual images. **[3]**

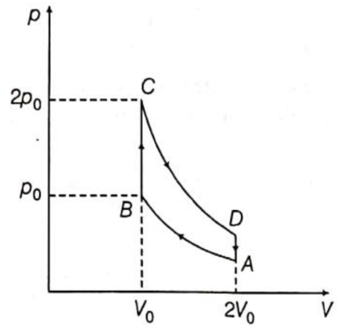
**3.** Two identical blocks $A$ and $B$ each of mass $M$ are placed on a long inclined plane (angle of inclination $= \theta$) with $A$ higher up than $B$. The coefficients of friction between the plane and the blocks $A$ and $B$ are respectively $\mu_A$ and $\mu_B$ with $\tan\theta > \mu_B > \mu_A$. The two blocks are initially held fixed at a distance $d$ apart. At $t = 0$ the two blocks are released from rest.

(a) At what time $t_1$ will the two blocks collide? **[2]**

(b) Consider each collision to be elastic. At what time $t_2$ and $t_3$ will the blocks collide a second and third time respectively? **[4]**

(c) Draw a schematic velocity-time diagram for the two blocks from $t = 0$ till $t = t_3$. Draw below them on a single diagram and use solid line (——) to depict block $A$ and dashed line (----) to depict block $B$. **[5]**

**4.** One mole of an ideal gas ($C_p / C_V = \gamma$ where symbols have their usual meanings) is subjected to an Otto cycle $ABCD$ as shown in the following $pV$ diagram. Path $AB$ and $CD$ are adiabats. The temperature at $B$ is $T_B = T_0$. Diagram is not to scale.

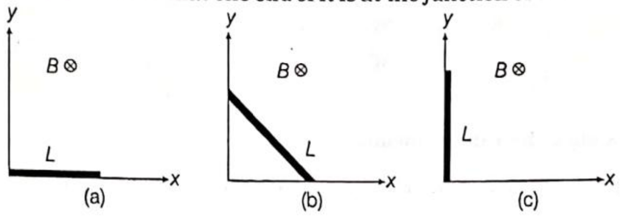



(a) Find the temperatures at $A$, $C$ and $D$ in terms of $T_0$ and pressures at $A$ and $D$ in terms of $p_0$. **[4]**

(b) Find total heat absorbed ($\Delta Q$) by the system, the total work done ($\Delta W$) and efficiency ($\eta$) of the Otto cycle in terms of $\gamma$ and related quantities. **[3½]**

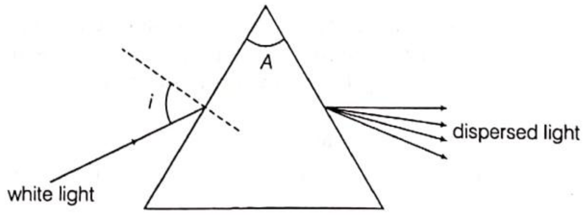
(c) Draw below corresponding $p$-$T$ and $T$-$S$ (entropy) diagrams for the cycle. **[6½]**

**5.** A metallic rod of mass $m$ and length $L$ (thick line in the figure below) can slide without friction on two perpendicular wires (thin lines in the figures). Entire arrangement is located in the horizontal plane. A constant magnetic field of magnitude $B$ exists perpendicular to this plane in the downward direction. The wires have negligible resistance compared to the rod whose resistance is $R$. Initially, the rod is along one of the wires so that one end of it is at the junction of the two wires (see Fig. (a)).




The rod is given an initial angular speed $\omega$ such that it slides with its two ends always in contact with the two wires (see Fig. (b)), and just comes to rest in an aligned position with the other wire (see fig. (c)). Determine the value of $\omega$. Neglect the self-inductance of the system. **[15]**

**6.** White light is incident at an angle $i$ on a prism of angle $A$ placed in air as shown. Let $D$ be the angular deviation (not necessarily a minimum) suffered by an emergent ray of a particular wavelength.




(a) Obtain an expression for $\sin(D + A - i)$ in terms of the refractive index $n$ and trigonometric functions of $i$ and $A$ only. **[3]**

(b) Let $A = 60.00°$ and $i = 45.62°$. Obtain the refractive index $(n_\lambda)$ for a ray of wavelength $\lambda$ which has suffered deviation $D = 49.58°$. **[2]**

(c) A detailed microscopic theory yields the relation between the refractive index $n$, of the material of the prism and the angular frequency $\omega = 2\pi c/\lambda$ of the incident light as

$$\frac{n^2 - 1}{n^2 + 2} = \frac{Ne^2}{3\varepsilon_0 m_e}\left(\frac{1}{\omega_0^2 - \omega^2}\right)$$

Here, $N$ is the electron density and $\omega_0 = 2\pi c/\lambda_0$ the natural frequency of oscillation of the electron of the material. The other symbols have their usual meaning. The table below lists the refractive indices at six wavelengths.

| $\lambda$(nm) | 706.54 | 667.82 | 501.57 | 492.19 | 447.15 | 438.79 |
|---|---|---|---|---|---|---|
| $n$ | 1.6087 | 1.6108 | 1.6263 | 1.6277 | 1.6358 | 1.6376 |

Re-express the above equation to get a linear relationship in terms of $\beta = (n^2 + 2)/(n^2 - 1)$ and a suitable power of $\lambda$. Tabulate and plot so that you may obtain $N$ and $\omega_0$. **[10]**

(d) Calculate the values of $N$, $\omega_0$ from the graph you plotted. Which part of the electromagnetic spectrum does $\lambda_0$ belong to? **[4]**

(e) An X-ray of energy 1.000 keV is incident on the prism. If we write $n = 1 + \delta$, then obtain the numerical value of $\delta$ for this ray. **[3]**

(f) For the X-ray of the previous part let $i_c$ be the critical angle and $\theta_c = 90 - i_c$ be the corresponding grazing angle. Obtain the value of $\theta_c$. **[1]**

# Detailed Solutions

**1.** Given, Mass of star $= M$

Mass of black hole $= M_b$

Radius of orbit $= R$

Angular frequency $= \omega$

(a) Dimension of power $= [M^1L^2T^{-3}]$

Speed of light $c = [LT^{-1}]$

Universal gravitational constant $G = [M^{-1}L^3T^{-2}]$

Mass $M = [M^1]$

Radius $R = [L^1]$

Angular frequency $\omega = [T^{-1}]$

Given, $L_G = Kc^xG^yM^2R^4\omega^6$

For a formula to be correct, the dimensions of LHS must be equal to dimensions of RHS.

i.e., Dimensions of $L_G$ = Dimensions of $Kc^xG^yM^2R^4\omega^6$

$$[M^1L^2T^{-3}] = [M^0L^0T^0][LT^{-1}]^x [M^{-1}L^3T^{-2}]^y[M]^2[L]^4[T^{-1}]^6$$

$$[M^1L^2T^{-3}] = [M^{2-y}L^{x+3y+4}T^{-x-2y-6}]$$

On comparing both sides, we get

$$2 - y = 1 \quad \ldots\text{(i)}$$

$$x + 3y + 4 = 2 \quad \ldots\text{(ii)}$$

and $$-x - 2y - 6 = -3 \quad \ldots\text{(iii)}$$

From Eq. (i), we get

$$y = 1$$

Substituting $y = 1$ in Eq. (ii), we get

$$x = -5,$$

Therefore, formula $L_G = Kc^{-5}GM^2R^4\omega^6$

(b) Since, it is given that the orbit remains circular throughout and Newtonian mechanics holds.

Therefore,

Total mechanical energy $(E)$ = Kinetic energy of star + potential energy of star

$$E = \frac{1}{2}M\omega^2R^2 + \left(-\frac{GMM_b}{R}\right) \quad \ldots\text{(i)}$$

$\because$ Orbit is circular so, centrifugal force = gravitation force

$$M\omega^2R = \frac{GMM_b}{R^2} \quad \ldots\text{(ii)}$$

On putting the values in Eq. (i) from Eq. (ii),

we get $E = \dfrac{GMM_b}{2R} - \dfrac{GMM_b}{R} = -\dfrac{GMM_b}{2R}$

(c) Here energy of the system is decreasing w.r.t. time through gravitational waves radiations so radius of orbit of star decreases.

According to Kepler's law,

$$T^2 = \frac{4\pi^2R^3}{GM_b}$$

$$\Rightarrow \quad R^3 = \frac{GT^2M_b}{4\pi^2} \Rightarrow R = \left(\frac{GT^2M_b}{4\pi^2}\right)^{\frac{1}{3}}$$

$$\because \quad E = -\frac{GMM_b}{2R}$$

On putting the value of $R$ in above expression, we get

$$E = -\frac{GMM_b}{2\left(\frac{GT^2M_b}{4\pi^2}\right)^{\frac{1}{3}}} = -\frac{(GM_b2\pi)^{\frac{2}{3}}M}{2T^{2/3}}$$

Power radiated $L_G$ = rate of decrease in total mechanical energy

i.e., $$L_G = -\frac{dE}{dt} = \frac{(GM_b2\pi)^{\frac{2}{3}}M}{2}\cdot\frac{d}{dt}\left(\frac{1}{T^{2/3}}\right)$$

$$L_G = \frac{(GM_b2\pi)^{\frac{2}{3}}M}{3T^{5/3}}\cdot\frac{dT}{dt} \quad \ldots\text{(iii)}$$

and $$L_G = Kc^{-5}GM^2R^4\omega^6 \quad \ldots\text{(iv)}$$

On equating Eqs. (iii) and (iv), we get

$$\frac{(GM_b2\pi)^{\frac{2}{3}}M}{3T^{5/3}}\frac{dT}{dt} = Kc^{-5}GM^2R^4\omega^6$$

$$\Rightarrow \quad \frac{dT}{dt} = \frac{3Kc^{-5}GM^2R^4\omega^6T^{5/3}}{(GM_b2\pi)^{2/3}M}$$

$$\because \quad \omega = \frac{2\pi}{T} \text{ and } R = \left(\frac{GT^2M_b}{4\pi^2}\right)^{\frac{1}{3}}$$

$$\Rightarrow \quad \frac{dT}{dt} = \frac{3Kc^{-5}GM^2T^{5/3}}{(GM_b2\pi)^{2/3}M}\left(\frac{2\pi}{T}\right)^6\cdot\left(\frac{GT^2M_b}{4\pi^2}\right)^{4/3}$$

On simplifying, we get

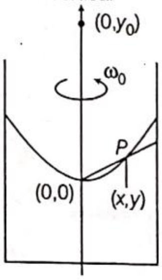
$$\frac{dT}{dt} = 3Kc^{-5}G^{1+\frac{4}{3}-\frac{2}{3}}MT^{\frac{5}{3}+\frac{8}{3}-6}M_b^{\frac{4}{3}-\frac{2}{3}}(2\pi)^{6-\frac{8}{3}-\frac{2}{3}}$$

$$= 3Kc^{-5}G^{5/3}\cdot M\cdot T^{-5/3}M_b^{\frac{2}{3}}(2\pi)^{8/3}$$

On rearranging, we get

$$\frac{dT}{dt} = \frac{3KG^{5/3}M_b^{\frac{2}{3}}M(2\pi)^{8/3}}{c^5T^{5/3}}$$

**2.** (a) Let us consider a point $P(x, y)$ on parabolical surface making an angle $\theta$ with vertical axis as shown in figure.




At this two acceleration acts one is centripetal acceleration (towards centre) and second is gravitational acceleration ($g$) (in negative $y$-axis)

$$\therefore \quad \tan\theta = \frac{a_x}{a_y} = \frac{\omega_0^2 x}{g} \quad \ldots(i)$$

$$\because \quad \tan\theta = \frac{dy}{dx} \quad \ldots(ii)$$

From Eqs. (i) and (ii), we get

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\omega_0^2 x}{g}$$

Differentiating w.r.t. '$x$', $\dfrac{d^2y}{dx^2} = \dfrac{\omega_0^2}{g}$

Given that radius of curvature of paraboloidal surface,

$$\rho = \frac{[1 + (dy/dx)^2]^{3/2}}{\dfrac{d^2y}{dx^2}}$$

On putting the values of $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in above expression, we get

$$\rho = \frac{\left[1 + \left(\dfrac{\omega_0^2 x}{g}\right)^2\right]^{3/2}}{\dfrac{\omega_0^2}{g}}$$

(b) At lowest point of Hg surface $x = 0$

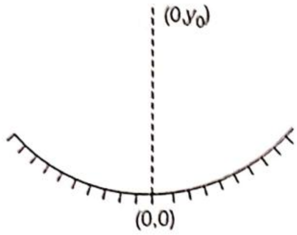
and $\omega_0 = 78$ rpm or 8.16 rad/s

$$\because \quad \rho = \frac{\left[1 + \left(\dfrac{\omega_0^2 x}{g}\right)^2\right]^{3/2}}{\omega_0^2 / g}$$

$$\rho_{x=0} = \frac{g}{\omega_0^2} = \frac{9 \cdot 8}{(8 \cdot 16)^2} = 0 \cdot 147\text{m}$$

or $\rho_{x=0} = 14.7$ cm

(c) As free surface of mercury is good reflecting surface i.e., it behaves as a curved reflecting surface which radius will $\rho_{x=0}$.




Using mirror formula $\dfrac{1}{f} = \dfrac{1}{v} + \dfrac{1}{u}$

Here $\quad f = -\dfrac{\rho_{x=0}}{2} = -\dfrac{g}{2\omega_0^2}$

$$u = -y_0, \ v = y_i$$

$$\therefore \quad -\frac{1}{f} = \frac{1}{y_i} - \frac{1}{y_0}$$

Putting the values in above expression

$$-\frac{2}{\rho_{x=0}} = \frac{1}{y_i} - \frac{1}{y_0}$$

$$\Rightarrow \quad -\frac{2\omega_0^2}{g} = \frac{1}{y_i} - \frac{1}{y_0}$$

$$\Rightarrow \quad \frac{1}{y_i} = \frac{1}{y_0} - \frac{2\omega_0^2}{g} = \frac{g - 2\omega_0^2 y_0}{g y_0}$$

$$\Rightarrow \quad y_i = \frac{g y_0}{g - 2\omega_0^2 y_0}$$

For real image $y_i$ should be negative i.e.,

$$g - 2\omega_0^2 y_0 < 0$$

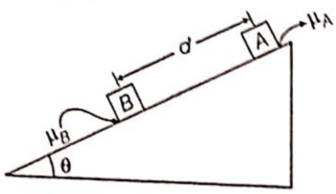
$$\Rightarrow \quad y_0 > \frac{g}{2\omega_0^2}$$

and for virtual image $y_i$ should be positive,

i.e., $\quad g - 2\omega_0^2 y_0 > 0$

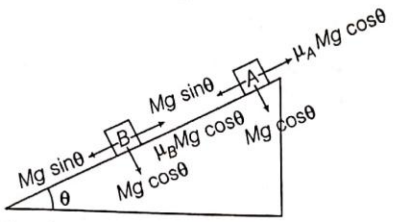
$$\Rightarrow \quad y_0 < \frac{g}{2\omega_0^2}$$

**3.** (a) Given, $\mu_A < \mu_B < \tan\theta$




Since, friction force acting on each block is less than pulling force $Mg\cos\theta$.

i.e., both blocks have some acceleration and also $\mu_A < \mu_B$ i.e., acceleration of block $A$ is greater than block $B$.




**Free body diagram**

Acceleration of block $B$,

$$a_B = \frac{\text{Pulling force} - \text{Opposing force}}{\text{Mass}}$$

$$= \frac{Mg\sin\theta - \mu_B Mg\cos\theta}{M}$$

$$a_B = g(\sin\theta - \mu_B\cos\theta)$$

Similarly, acceleration of block $A$,

$$a_A = g(\sin\theta - \mu_A\cos\theta)$$

Now, using relative motion concept, assume block $B$ at rest.

Acceleration of block $A$, w.r.t. block $B$,

$$a_{AB} = a_A - a_B = g(\mu_B - \mu_A)\cos\theta$$

Initially both are at rest so $u_{AB} = 0$, $s_{AB} = d$ and let $t_1$ is the time to collide then using second equation of motion

$$s = ut + \frac{1}{2}at^2$$

$$d = 0 + \frac{1}{2}g(\mu_B - \mu_A)\cos\theta \times t_1^2$$

$$\Rightarrow \quad t_1 = \sqrt{\frac{2d}{(\mu_B - \mu_A)g\cos\theta}}$$

(b) Let $v_A$ and $v_B$ be the velocities of block $A$ and block $B$ after $t_1$ time just before first collision

$$\therefore \quad v_A = u_A + a_A t_1 = 0 + g(\sin\theta - \mu_A\cos\theta)t_1$$

and $\quad v_B = g(\sin\theta - \mu_B\cos\theta)t_1$

In the elastic collision between two blocks, velocities are interchanged but acceleration remains same if mass of block are equal.

So, velocities of blocks just after first collision,

$$u_{A_1} = g(\sin\theta - \mu_B\cos\theta)t_1$$

and $\quad u_{B_1} = g(\sin\theta - \mu_A\cos\theta)t_1$

Velocity of block $A$ w.r.t. block $B$,

$$u_{A_1B_1} = (\mu_A - \mu_B)\,g\cos\theta t_1$$

Acceleration of block $A$, w.r.t. block $B$

$$a_{AB} = g(\mu_B - \mu_A)\cos\theta$$

If the time $t$ is for second collision just after first collision, then using second equation of motion

$$d_{AB} = u_{AB}t + \frac{1}{2}a_{AB}t^2$$

$$0 = -g(\mu_B - \mu_A)\cos\theta t_1 t + \frac{1}{2}g(\mu_B - \mu_A)\cos\theta t^2$$

$$\Rightarrow \frac{1}{2}g(\mu_B - \mu_A)\cos\theta t^2 = g(\mu_B - \mu_A)\cos\theta t_1 t$$

$$\Rightarrow \quad t = 2t_1$$

Total time for second collision $t_2 = t_1 + t = 3t_1$

If the velocity of block $A$ and block $B$ just before second collision be $v'_{A_1}$ and $v'_{B_1}$ respectively, then

$$v'_{A_1} = v_{A_1} + a_A t = g(\sin\theta - \mu_B\cos\theta)t_1 + g(\sin\theta - \mu_A\cos\theta)\,2t_1$$

$$= [3\sin\theta - \cos\theta(2\mu_A + \mu_B)]gt_1$$

$$v'_{B_1} = v_{B_1} + a_B t$$

$$= g[\sin\theta - \mu_A\sin\theta]t_1 + g(\sin\theta - \mu_B\cos\theta)\,2t_1$$

$$v'_{B_1} = [3\sin\theta - \cos\theta(\mu_A + 2\mu_B)]\,gt_1$$

After second elastic collision velocities of block will be interchanged. Let $u_{A_2}$ and $u_{B_2}$ be the velocities of blocks $A$ and $B$ just after second collision.

i.e., $\quad u_{A_2} = v'_{B_1}$

$$= [3\sin\theta - \cos\theta(\mu_A + 2\mu_B)]\,gt_1$$

and $\quad u_{B_2} = v'_{A_1} = [3\sin\theta - \cos\theta(2\mu_A + \mu_B)]\,gt_1$

Velocity of block $B$ w.r.t. block $A$

$$u_{A_2B_2} = gt_1\cos\theta(\mu_B - \mu_A) = 0$$

If $t$ is the time taken by two blocks, for third collision just after second collision then using second equation of motion.

$$d_{AB} = u_{A_2B_2}t + \frac{1}{2}a_{AB}t^2$$

$$0 = 0\cdot t_1 t + \frac{1}{2}g(\mu_B - \mu_A)\cos\theta\, t^2$$

$$\Rightarrow \quad t = 2t_1$$

From initial total time taken for third collision,

$$t_3 = t_2 + t$$

$$= 3t_1 + 2t_1 = 5t_1$$

Similarly, they will collide elastically and velocity will interchange and so on.

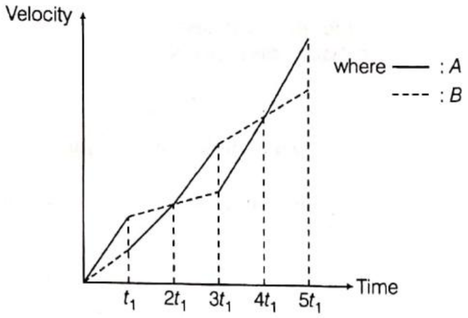
(c) For first collision, time $t_1$ and $v_A > v_B$

For second collision, time $3t_1$ (total)

and $\quad v_{A_1} < v_{A_2}$

For third collision, time $3t_2$ (total)

and $\quad v_{A_2} > v_{A_1}$




If the $t$ is time after first collision when velocity of block becomes equal then, using first equation of motion.

$$v'_A = v'_B$$

$$u_{A_1} + a_A t = u_{B_1} + a_B t$$

$$g(\sin\theta - \mu_B\cos\theta)t_1 + g(\sin\theta - \mu_A\cos\theta)t$$
$$= g(\sin\theta - \mu_A\cos\theta)t_1 + g(\sin\theta - \mu_B\cos\theta)t$$

$$g[\sin\theta - \mu_B\cos\theta - \sin\theta + \mu_A\cos\theta]t$$
$$= g[\mu_A\cos\theta - \mu_B\cos\theta]t_1$$

$$\Rightarrow \quad t = t_1$$

So $t_1 + t(2t_1)$ is from initial where velocity of both blocks becomes equal.

Again if $t$ is time of second collision when velocities of two blocks become equal, using first equation of motion

$$v''_A = v''_B$$

$$u_{A_2} + a_A t = u_{B_2} + a_B t$$

$$[3\sin\theta - \cos\theta(\mu_A + 2\mu_B)]gt_1 + g[\sin\theta - \mu_A\cos\theta)]t$$
$$= [3\sin\theta - \cos\theta(2\mu_A + \mu_B)]gt_1 + g[\sin\theta - \mu_B\cos\theta]t$$

On solving, we get

$$t = t_1$$

So total time $3t_1 + t_1 (= 4t_1)$ taken from initial after second collision where velocities are again equal.

Similarly, $6t_1$, $8t_1$, ... are the time from initial after third collision, fourth collision, ... where velocity is found to be equal.

**4.** (a) Since, $AB$ process is adiabatic.

State equation $TV^{\gamma-1}$ = constant

$$\Rightarrow \quad \frac{T_A}{T_B} = \left(\frac{V_B}{V_A}\right)^{\gamma-1} \Rightarrow \frac{T_A}{T_0} = \left(\frac{V_0}{2V_0}\right)^{\gamma-1}$$

$$\Rightarrow \quad T_A = \frac{T_0}{2^{\gamma-1}} \qquad \ldots(i)$$

Again using state equation of adiabatic process, $pV^\gamma$ = constant

$$\frac{p_A}{p_B} = \left(\frac{V_B}{V_A}\right)^\gamma \Rightarrow \frac{p_A}{p_0} = \left(\frac{V_0}{2V_0}\right)^\gamma$$

$$\Rightarrow \quad p_A = \frac{p_0}{2^\gamma} \qquad \ldots(ii)$$

**For process *BC*** Process is isochoric.

From state equation of isochoric process,

$$\frac{p}{T} = \text{constant}$$

$$\Rightarrow \quad \frac{T_C}{T_B} = \frac{p_C}{p_B} \Rightarrow \frac{T_C}{T_0} = \frac{2p_0}{p_0}$$

$$\Rightarrow \quad T_C = 2T_0 \qquad \ldots(iii)$$

**For process *CD*** Process is adiabatic using state equation.

$$TV^{\gamma-1} = \text{constant}$$

$$\frac{T_D}{T_C} = \left(\frac{V_C}{V_D}\right)^{\gamma-1} \Rightarrow \frac{T_D}{2T_0} = \left(\frac{V_0}{2V_0}\right)^{\gamma-1}$$

$$\Rightarrow \quad T_D = \frac{T_0}{2^{\gamma-2}} \qquad \ldots(iv)$$

Again, using state equation,

$$pV^\gamma = \text{constant}$$

$$\frac{p_D}{p_C} = \left(\frac{V_C}{V_D}\right)^\gamma \Rightarrow \frac{p_D}{2p_0} = \left(\frac{V_0}{2V_0}\right)^\gamma$$

$$\Rightarrow \quad p_D = \frac{p_0}{2^{\gamma-1}} \qquad \ldots(v)$$

(b) Heat absorbed by system $\Delta Q$ = sum of heat absorbed in all four processes

$$\Delta Q = (\Delta Q)_{AB} + (\Delta Q)_{BC} + (\Delta Q)_{CD} + (\Delta Q)_{DA}$$

Here processes $AB$ and $CD$ are adiabatic and in adiabatic process, $\Delta Q = 0$

$$\therefore \quad \Delta Q = (\Delta Q)_{BC} + (\Delta Q)_{DA}$$
$$= \mu C_V(T_C - T_B) + \mu C_V(T_A - T_D)$$

(Because at constant volume, given heat is equal to $\mu C_V \Delta T$)

From Eqs. (i), (iii) and (iv)

$$\Delta Q = \mu C_V(T_C - T_B + T_A - T_D)$$
$$= 1 \times C_V\left(2T_0 - T_0 + \frac{T_0}{2^{\gamma-1}} - \frac{T_0}{2^{\gamma-2}}\right)$$
$$= C_V\left(T_0 - \frac{T_0}{2^{\gamma-1}}\right)$$

$$\Delta Q = C_V T_0\left(1 - \frac{1}{2^{\gamma-1}}\right)$$

For a closed cycle, the change in internal energy is zero i.e., $dU = 0$

From first law of thermodynamics

$$\Delta Q = dU + \Delta W$$

$$\Delta Q = 0 + \Delta W$$

$$\Rightarrow \quad \Delta W = \Delta Q = C_V T_0\left(1 - \frac{1}{2^{\gamma-1}}\right)$$

Efficiency, $\eta = 1 - \frac{Q_{DA}}{Q_{BC}}$

$$= 1 - \frac{\left[\text{Heat released during isochoric process } DA\right]}{\left[\text{Heat absorbed during isochoric process } BC\right]}$$

$$\eta = 1 - \frac{\mu C_V(T_D - T_A)}{\mu C_V(T_C - T_B)} = 1 - \frac{\frac{T_0}{2^{\gamma-2}} - \frac{T_0}{2^{\gamma-1}}}{2T_0 - T_0}$$

$$= 1 - \left(\frac{1}{2^{\gamma-2}} - \frac{1}{2^{\gamma-1}}\right) = 1 - \frac{1}{2^{\gamma-1}}(2-1)$$

$$\eta = 1 - \frac{1}{2^{\gamma-1}}$$

(c) Process $AB$ is an adiabatic process with

$$p_B > p_A$$

$$\Rightarrow \quad T_B > T_A$$

***p-T* graph**, from state equation,

$$p^{\gamma}T^{\gamma-1} = \text{constant}$$

So, $p$-$T$ graph will not be straight line.

***T-S* graph**, since, $(\Delta Q)_{AB} = 0$

$$\Rightarrow \quad (\Delta S)_{AB} = 0$$

But $\quad T_B > T_A$

Process $BC$ is an isochoric process with

$$p_C > p_B$$

$$\Rightarrow \quad T_C > T_B$$

***p-T* graph**, $V$ = constant, therefore, $p$-$T$ graph is a straight line passing through origin

***T-S* graph**, $(\Delta Q)_{BC} = \mu C_V dT$ = positive

$$dS = \int \frac{dQ}{T}$$

$$= \mu C_V \int_T^T \frac{dT}{T} = \mu C_V \ln\left(\frac{T_C}{T_B}\right)$$

Entropy increases logarithmically as $T$ increases $[\because T > T_B]$

Process $CD$ is an adiabatic process $p_D < p_C$

$$\Rightarrow \quad T_D < T_C \quad \text{and} \quad (\Delta Q)_{CD} = 0$$

$$\Rightarrow \quad (\Delta S)_{CD} = 0$$

Process $DA$ is an isochoric process with

$$p_A < p_D$$

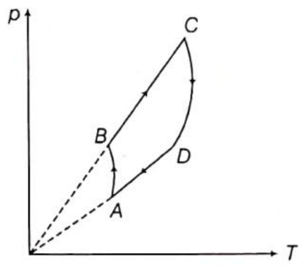
$$\Rightarrow \quad T_A < T_D$$

$$(\Delta Q)_{DA} = \mu C_v dT = \text{negative}$$

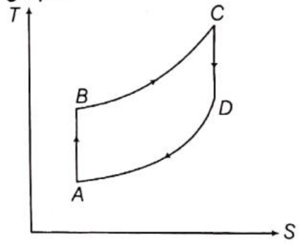
$$dS = \mu C_V \ln\left(\frac{T}{T_A}\right)$$

Entropy decreases logarithmically as $T$ increases. $[\because T > T_A]$

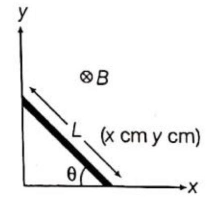
***p-T* graph**




***T-S* graph**




**5.** Let us consider rod is making an angle $\theta$ from the horizontal axis after giving initial angular speed $\omega$ (before come to rest) as shown in adjoining figure.




Let the coordinates of the centre of mass of the rod $(x_{cm}, y_{cm})$.

$$x_{cm} = \frac{L}{2}\cos\theta \text{ and } y_{cm} = \frac{L}{2}\sin\theta$$

At any instant during motion, the kinetic energy possessed by rod

$K$ = translating kinetic energy + Rotational kinetic energy

$$= \frac{1}{2}mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2 \quad \ldots(i)$$

where, $I_{cm}$ = moment of inertia of rod about an axis which passes through centre of mass and perpendicular to horizontal plane.

i.e., $$I_{cm} = \frac{mL^2}{12} \quad \text{...(ii)}$$

On putting in Eq. (i),

$$K = \frac{1}{2}mv_{cm}^2 + \frac{1}{2}\cdot\frac{mL^2}{12}\omega^2$$

$$= \frac{1}{2}m\left(\frac{L}{2}\right)^2 + \frac{1}{2}\frac{mL^2}{12}\omega^2 \quad [\because v_{cm} = \omega r]$$

$$K = \frac{mL^2\omega^2}{6} \quad \text{...(iii)}$$

Since, a conducting rod is kept in uniform magnetic field which is initially at rest. After giving angular speed, it forms a conducting loop with the help of wire whose area varies as metallic rod moves.

When flux associated to a conducting loop varies with respect to time, a induced emf is generated which is equal to rate of change of flux.

i.e., $$\varepsilon = \frac{d\phi}{dt} \quad \text{... (iv)}$$

$$= \frac{d}{dt}(BA) = B\frac{dA}{dt} \quad [\because B \text{ is uniform}]$$

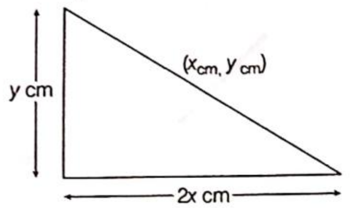
Area of loop at any time

$$A = \frac{1}{2}\times \text{base} \times \text{height} = \frac{1}{2}\times(2x_{cm})\times(2y_{cm})$$

$$A = 2x_{cm}y_{cm}$$

$$= 2\left(\frac{L}{2}\cos\theta\right)\times\left(\frac{L}{2}\sin\theta\right) = \frac{L^2}{2}\sin\theta\cos\theta$$

$$A = \frac{L^2}{4}\sin 2\theta \quad \text{...(v)}$$




$$\Rightarrow \quad \frac{dA}{d\theta} = \frac{L^2}{2}\cos 2\theta \Rightarrow dA = \frac{L^2}{2}\cos 2\theta d\theta$$

Putting this in Eq. (iv),

$$\Rightarrow \quad \frac{dA}{dt} = \frac{\omega L^2}{2}\cos 2\theta \Rightarrow \varepsilon = \frac{B\omega L^2}{2}\cos 2\theta$$

The resistance of loop is $R$ and power consumed by circuit is equal to $\frac{\varepsilon^2}{R}$

The power dissipated due to the current is equal to the loss of kinetic energy or rod i.e.,

$$-\frac{dK}{dt} = \frac{\varepsilon^2}{R}$$

$$\Rightarrow \quad dK = -\frac{\varepsilon^2}{R}dt = -\frac{1}{R}\left(\frac{B\omega L^2}{2}\cos 2\theta\right)^2 dt$$

$$\Rightarrow \quad dK = -\frac{B^2\omega^2L^4}{4R}\cos^2 2\theta\, dt$$

Integrating on both sides,we get

$$\Rightarrow \quad \int_{K_i = K}^{K_f = 0} dK = -\frac{B^2\omega^2L^4}{4R}\int_{\theta=0}^{\theta=\frac{\pi}{2}}\cos^2 2\theta dt$$

$$\Rightarrow \quad [K]_{K_i = K}^{K_f = 0} = -\frac{B^2\omega^2L^4}{4R}\int_{\theta=0}^{\theta=\frac{\pi}{2}}\frac{1+\cos 4\theta}{2}dt$$

$$\Rightarrow \quad 0 - K = -\frac{B^2\omega^2L^4}{4R}\int_{\theta=0}^{\theta=\frac{\pi}{2}}\frac{1+\cos 4\theta}{2}\cdot\frac{d\theta}{\Omega}$$

$$\Rightarrow \quad -K = -\frac{B^2\omega L^4}{8R}\int_{\theta=0}^{\theta=\frac{\pi}{2}}(1+\cos 4\theta)d\theta$$

$$\Rightarrow \quad K = \frac{B^2\omega L^4}{8R}\left\{\theta + \frac{\sin 4\theta}{4}\right\}_{\theta=0}^{\theta=\frac{\pi}{2}}$$

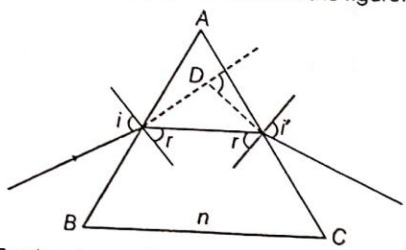
$$\Rightarrow \quad K = \frac{B^2\omega L^4}{8R}\left\{\left(\frac{\pi}{2} - 0\right) + \frac{1}{4}(\sin 2\pi - \sin 0)\right\}$$

$$\Rightarrow \quad k = \frac{B^2\Omega L^4}{8R}\cdot\frac{\pi}{2}$$

On putting the value of $K$ from Eq. (iii)

$$\frac{mL^2l^2}{6} = \frac{B^2\omega L^4}{8R}\cdot\frac{\pi}{2} \Rightarrow \omega = \frac{3\pi B^2L^2}{8mR}$$

**6.** (a) Consider the situation shown in the figure.




For the above prism, we can write

$$D = i + i' - A$$

Here, symbols have their usual meaning.

$$\Rightarrow \quad D + A - i = i'$$

$$\Rightarrow \quad \sin(D + A - i) = \sin i' \quad \text{...(i)}$$

Again, we have $r + r' = A$

$$\Rightarrow \quad r' = A - r \quad \text{...(ii)}$$

Now, from Snell's law, $n \sin r' = \sin i'$

$\Rightarrow \quad n\,[\sin(A - r)] = \sin i'$

$\Rightarrow \quad \sin i' = n\,[\sin A \cdot \cos r - \cos A \cdot \sin r]$

$\Rightarrow$

$\sin i' = n\,[\sin A \cdot \sqrt{1 - \sin^2 r} - \cos A \cdot \sin r]$

$$= n\left[\sin A\sqrt{1 - \frac{\sin^2 i}{n^2}} - \cos A \cdot \frac{\sin i}{n}\right]$$

$$\Rightarrow \quad \sin i' = [\sin A\sqrt{n^2 - \sin^2 i} - \sin i \cos A] \quad \text{...(iii)}$$

Now, from Eqs. (i) and (iii), we have

$\sin(D + A - i)$

$$= [\sin A\sqrt{n^2 - \sin^2 i} - \sin i \cos A] \quad \text{...(iv)}$$

(b) Now, it is given that

$A = 60.00°, i = 45.62°, D = 49.58°$

Putting these values in Eq. (iv), we get

$\sin(49.58 + 60.00 - 45.62)$

$$= \sin(60.00)\sqrt{n_\lambda^2 - \sin^2(45.62)} - \sin(45.62)\cos(60.00)$$

$$\Rightarrow \sin(63.96) = \frac{\sqrt{3}}{2}\sqrt{n_\lambda^2 - (0.51)} - 0.71 \times \left(\frac{1}{2}\right)$$

$$\Rightarrow \quad 0.89 = 0.86\sqrt{n^2\lambda - 0.51} - 0.355$$

$$\Rightarrow \quad 1.44 = \sqrt{n_\lambda^2 - 0.51}$$

$$\Rightarrow \quad n_\lambda^2 = 2.605 \Rightarrow n_\lambda = 1.61$$

Putting the values of $n$ and $\lambda$, we will obtain the table as shown below.

| $\lambda$(nm) | $n$ | $\frac{1}{\lambda^2}(\times 10^{-6}\text{mm}^{-2})$ | $\beta = \frac{n^2+2}{n^2-1}$ |
|---|---|---|---|
| 706.54 | 1.6087 | 2.0032 | 2.8893 |
| 667.82 | 1.6108 | 2.2422 | 2.8813 |
| 501.57 | 1.6263 | 3.9750 | 2.8239 |
| 492.19 | 1.6277 | 4.1280 | 2.8188 |
| 447.15 | 1.6358 | 5.0014 | 2.7901 |
| 438.79 | 1.6376 | 5.1938 | 2.7839 |

(c) Given, $\dfrac{n^2 - 1}{n^2 + 2} = \dfrac{Ne^2}{3\varepsilon_0 m_e}\left(\dfrac{1}{\omega_0^2 - \omega^2}\right)$

Also, it is given that

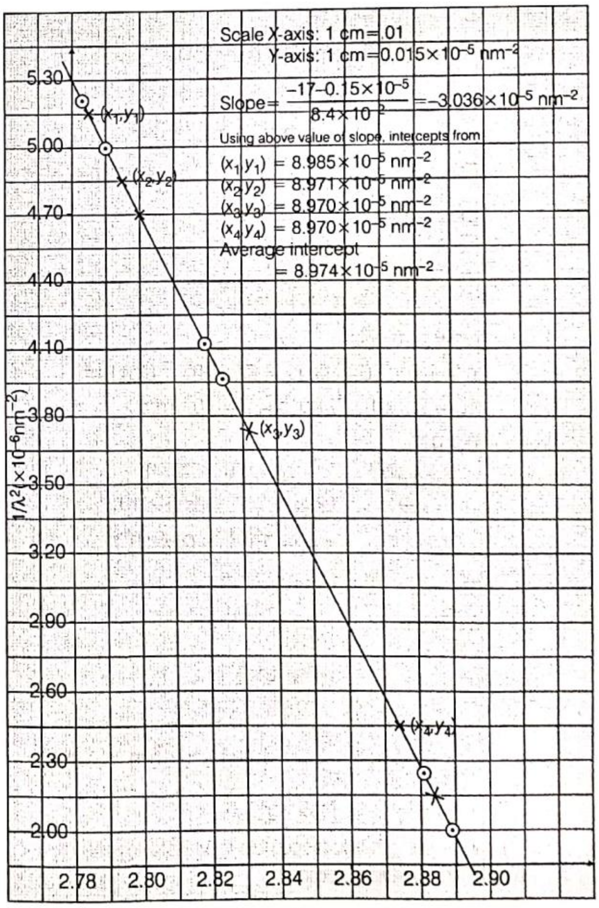
$$\beta = \frac{n^2 + 2}{n^2 - 1} \Rightarrow \frac{1}{\beta} = \frac{Ne^2}{3\varepsilon_0 m_e}\left(\frac{1}{\omega_0^2 - \omega^2}\right)$$

$$\Rightarrow \quad \frac{1}{\beta} = \frac{Ne^2}{3\varepsilon_0 m_e}\left[\frac{1}{\left(\frac{2\pi c}{\lambda_0}\right)^2 - \left(\frac{2\pi c}{\lambda}\right)^2}\right]$$

$$\Rightarrow \quad \frac{1}{\beta} = \frac{Ne^2}{12\pi^2\varepsilon_0 m_e c^2}\left[\frac{\lambda_0^2\lambda^2}{\lambda^2 - \lambda_0^2}\right]$$

$$\Rightarrow \quad \beta = \frac{12\pi^2\varepsilon_0 m_e c^2}{Ne^2}\left[\frac{\lambda^2 - \lambda_0^2}{\lambda_0^2\lambda^2}\right] \quad \text{...(v)}$$

From the above table, the plot between $\beta$ and $\frac{1}{\lambda^2}$, can be drawn as shown below.




(d) Now, putting $\beta = 0$, in Eq. (v), we get

$$\lambda^2 - \lambda_0^2 = 0$$

$$\Rightarrow \quad \frac{1}{\lambda^2} = \frac{1}{\lambda_0^2}$$

$$\Rightarrow\ 5.30 \times 10^{-6}\ \text{nm}^{-2} = \frac{1}{\lambda_0^2}$$

$$\Rightarrow\ \frac{1}{\lambda_0} = \sqrt{5.3} \times 10^{-3}\ \text{nm}^{-1}$$

$$\Rightarrow\ \frac{2\pi}{\lambda_0} = 2\pi \times \sqrt{5.3} \times 10^{-3} \times 10^{+9}$$

$$\Rightarrow\ \omega_0 = 2\pi \times \sqrt{5.3} \times 10^{-3} \times 10^{+9}$$

$$= 14.45 \times 10^{-3} \times 10^{9}$$

$$= 14.45 \times 10^{6}\ \text{rad/s}$$

Now, Eq. (v) can be expressed as

$$\frac{1}{\lambda^2} = \frac{1}{\lambda_0^2} - \frac{\beta N e^2}{12\pi^2 \varepsilon_0 m_e c^2}$$

Slope of the $\frac{1}{\lambda^2}$ *versus* $\beta$ graph is

$$\text{Slope} = \frac{Ne^2}{12\pi^2 \times \varepsilon_0 m_e c^2}$$

From the graph,

$$\text{Slope} = \frac{5.3 \times 10^{-6} \times 10^{18}}{2.9}$$

$$= \frac{N \times e^2}{12 \times \pi^2 \times \varepsilon_0 \times m_e \times c^2}$$

Now putting,

$e = 1.6 \times 10^{-19}$ C, $\varepsilon_0 = 8.8 \times 10^{-12}\ C^2/\text{H-m}^2$

$m_e = 9.1 \times 10^{-31}$ kg, $c = 3 \times 10^{8}$ m/s

We get,

$$\frac{5.3 \times 10^{12}}{2.9} = \frac{N \times (1.6 \times 10^{-19})^2}{12 \times (3.14)^2 \times 8.85 \times 10^{-12} \times 9.1 \times 10^{-31} \times (3 \times 10^{8})^2}$$

$$\Rightarrow\ N = 1.01 \times 10^{29}\ /\text{m}^3$$

We have, $\lambda_0 = \frac{1}{\sqrt{5.3}} \times 10^{3}$ nm

$$= \frac{10^{3} \times 10^{-9}}{\sqrt{5.3}}$$

$$= \frac{10^{-6}}{\sqrt{5.3}} = 0.434 \times 10^{-6}\ \text{m}$$

$$= 4.3 \times 10^{-7}\ \text{m}$$

This wavelength corresponds to visible region of electromagnetic spectrum.

(e) Given that,

Energy of X-ray, $E = 1000$ eV

Speed of light, $c = 3 \times 10^{8}$ m/s

$h = 6.63 \times 10^{-34}$ J-s

$e = 1.6 \times 10^{-19}$ C

$$e = \frac{hc}{\lambda}$$

$$\Rightarrow\ \lambda = \frac{hc}{E}$$

$$\lambda = \frac{6.63 \times 10^{-34} \times 3 \times 10^{8}}{1000 \times 1.6 \times 10^{-19}}$$

$$= 12.43125 \times 10^{-10}\text{m}$$

$$N = 1.01 \times 10^{29}\ /\text{m}^3$$

$$m_e = 9.11 \times 10^{-31}\text{kg}$$

$$\omega_0 = 1.78 \times 10^{16}\ \text{Hz} = \frac{1.78}{2\pi} \times 10^{6}\ \text{rad/s}$$

$$\varepsilon_0 = \frac{10^{-9}}{36\pi}$$

$$\omega = \frac{c}{\lambda} = \frac{3 \times 10^{8}}{12.43125 \times 10^{-10}}$$

$$= 24.13273 \times 10^{16}\ \text{rad/s}$$

Now, using formula,

$$\frac{n^2 - 1}{n^2 + 2} = \frac{Ne^2}{3\varepsilon_0 m_e}\left(\frac{1}{\omega_0^2 - \omega^2}\right)$$

$$= \frac{1.01 \times 10^{29} \times (1.6 \times 10^{-19})^2}{3 \times \frac{10^{-9}}{36\pi} \times 9.11 \times 10^{-31}} \left[\frac{1}{\left(\frac{1.78}{2\pi} \times 10^{16}\right)^2 - (24.13273 \times 10^{16})^2}\right]$$

$$= \frac{1.01 \times 1.6 \times 1.6 \times 36\pi \times 31 \times 10}{3 \times 9.11} \left[\frac{1}{582.3084}\right] \times \frac{1}{10^{32}}$$

$$\frac{n^2 - 1}{n^2 + 2} = -\,1.837 \times 10^{-3}$$

$$n^2 - 1 = -\,1.837 \times 10^{-3}[n^2 + 2]$$

$$n^2[1 + 1.837 \times 10^{-3}] = 1 - 3.675 \times 10^{-3}$$

$$n^2[1.001837] = 0.996325$$

$$n^2 = 0.994498 \Rightarrow n = 0.99724$$

Given, $n = 1 + \delta \Rightarrow \delta = n - 1 = 0.994498 - 1$

$$\delta = -\,5.5 \times 10^{-3}\ \text{rad}$$

(f) $i_c \rightarrow$ critical angle and $\theta_c \rightarrow$ grazing angle

$$i_c = \sin^{-1}\left[\frac{\text{Refractive index of medium for X-ray}}{\text{Refractive index of air}}\right]$$

$$i_c = \sin^{-1}\left(\frac{0.99993}{1}\right) = 89.322°$$

$$\theta_c = 90° - i_c = 90° - 89.322° = 0.677° \simeq 0.68°$$